JOURNAL OF ALGEBRA 126, 400–450 (1989)

Cohomology of Nilpotent Groups of Class 2

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Communicated by Saunders Mac Lane

Received August 13, 1987

We compute the cohomology rings of a number of nilpotent groups of class 2 for appropriate coefficients, and we do some more sample calculations of various cohomology groups. The basic tool is a suitable small free resolution for an arbitrary nilpotent group of class 2 constructed in a previous paper.

INTRODUCTION

This is the *second* of a series of papers about the cohomology of solvable groups. In [49] we constructed a small free resolution for an arbitrary nilpotent group G of class 2 by means of a perturbation which reflects the structure of G as a central extension of an abelian group by an abelian group, thereby turning what might appear at first glance as a nuisance to one's advantage. We indicate the dependence of the present paper on [49] by continuing the numbering of sections and that of the bibliography. Also the notation will be the same as in [49], and any unexplained notation may be found there.

In the present paper we use the free resolution obtained in [49] to compute a number of examples. We cannot resist giving these examples; we wish to justify our approach, and calculation seems to be the way to achieve this goal. We apologize for the length of the paper, which is to a great extent due to the obstacles we encountered in carrying out the calculations. In retrospect, the amount of labor involved was rather prohibitive.

We now explain the contents of the paper. In Section 5 we reproduce briefly the construction of our small free resolution. In Section 6 we examine free nilpotent groups of class 2. These are universal examples in an

* Supported by the Deutsche Forschungsgemeinschaft under a Heisenberg grant.

0021-8693/89 $3.00

appropriate sense. In Section 7 we study nilpotent groups of class two whose quotient modulo a central subgroup is free abelian, and in Section 8 we investigate the general case, i.e., groups of class 2 that cannot necessarily be written as a central extension with free abelian quotient. It turns out that the general case is much harder to deal with than the special ones studied in Sections 6 and 7. We illustrate our approach with a number of examples. One such example which we shall study is the group

$$G = \langle x_1, x_2, y; [x_2, x_1] = y, [x_1, y] = 1, [x_2, y] = 1, x_1^m = 1, y^l = 1 \rangle,$$

where l and m are numbers so that l divides m; a small free resolution for it will be obtained in (8.3). It is clear that we then have the machinery in place to carry out the calculations of the homology and cohomology of G with arbitrary coefficients. For brevity, we compute its integral cohomology ring in a number of special cases which are phrased in terms of suitable number theory relations between the numbers l and m. At this stage we only mention that in the general case one quickly runs into non-trivial number theory problems related with invariant theory; see (8.7) below. For our purposes this example is the simplest one of a central extension of an abelian group by an abelian group which are both not free. A different approach to this example would be to write G as a semidirect product $G = (\mathbf{Z}/l \times \mathbf{Z}/m) \rtimes \mathbf{Z}$. In the standard way, this implies that for any G-module M, $\mathrm{H}^*(G, M)$ fits into a short exact sequence

$$0 \to \mathrm{H}^1(\mathbf{Z}, \mathrm{H}^{*-1}(\mathbf{Z}/l \times \mathbf{Z}/m, M)) \to \mathrm{H}^*(G, M)$$
$$\to \mathrm{H}^0(\mathbf{Z}, \mathrm{H}^*(\mathbf{Z}/l \times \mathbf{Z}/m, M)) \to 0$$

but in general this determines $\mathrm{H}^*(G, M)$ only up to an extension. As far as the present example is concerned, it is precisely at this stage that our approach pays off: It enables us to solve the extension problem, and we shall show that in general it is in fact non-trivial. Also it is worthwhile emphasizing that the significance of this example is that we have a general construction which, if applied to this example, yields a small free resolution for it and hence its homology and cohomology with arbitrary coefficients.

In a similar vein, in (8.10) we apply our approach to finite groups of the kind

$$G(m_1, m_2, l) = \langle x_1, x_2, y; [x_2, x_1] = y, [x_2, y] = 1,$$
$$[x_1, y] = 1, y^l = 1, x_1^{m_1} = 1, x_2^{m_2} = 1 \rangle,$$

where l divides the numbers m_1 and m_2. Although we have an explicit small free resolution for such groups, we were so far unable to compute the differential is a closed form in general, and in (8.10) we only offer a number of sample calculations. We intend to complete elsewhere the calculation of the differential in a closed form. However, interesting information can already be deduced from what we have obtained so far; we shall illustrate this in (8.10) below. The computation of more sophisticated examples than those given in the present paper leads to problems which are presumably more conveniently solved on a computer than by hand. Again we intend to return to this elsewhere.

The construction of our free resolution also casts new light on various well-known spectral sequences, including those for a group extension and spectral sequences of the Eilenberg–Moore type. These spectral sequences play no role in the present paper. We intend to explain elsewhere how these spectral sequences show up in our construction; they relate our approach to the more conventional ones in terms of these spectral sequences. However, it is worthwhile emphasizing that while the higher differentials in these spectral sequences have not been tractable in any reasonable way before since their calculation requires more than just formal properties of the spectral sequences, our free resolution enables us to compute higher differentials. Furthermore, our approach reveals among others that the higher differentials in the spectral sequence of a group extension are related with certain higher Massey products which reflect the structure of the extension; example will be given elsewhere. It seems that the lack of insight into such higher Massey products and the lack of methods to manipulate them partly explains why so little has been known about the homology and cohomology of central extensions, in particular of nilpotent groups of class 2. The perturbation theory given in [49] offers a way around this; it actually avoids these Massey products, but with hindsight, i.e., after the requisite perturbations have been calculated, one can also compute the higher Massey products that come into play.

5. Recollections

Let

$$\mathbf{e}\colon 0 \to N \to G \to Q \to 1$$

be a central extension with abelian quotient group Q, so that G is nilpotent of class at most 2, and let R be a commutative ring with 1. For intelligibility we recall briefly the constructions (4.2.6) and (4.3.7) in [49] of a small free resolution $\mathrm{M}(G)$ of R in the category of right RG-modules.

Extend the given central extension to a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & L \arrow[r, "\mathrm{Id}"] \arrow[d] & L \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{Id}"'] & G_F \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & N \arrow[r] & G \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd} \tag{5.1}$$

of central extensions as indicated, where the right-hand column is a free presentation of Q as an abelian group of the kind [49, (3.2.1.)], and where G_F is the pull back group; in particular, when Q is free abelian, we may take $F=Q$ and $L=0$, and we shall always do so. Now the acyclic construction (3.1.2) or (3.2.6) for RQ in [49] according as Q is or is not free abelian looks like

$$\mathrm{M}(Q)=\Gamma_L \otimes_{t_Q} (\Lambda_F \otimes_{\tau_F} RQ),$$

and its underlying chain complex is a free resolution of R in the category of RQ-modules. Here $\Lambda_F = \mathrm{H}_*(F)$, $\Gamma_L = \mathrm{H}_*(L, 2)$, and τ_F and t_Q are the twisting cochains (3.1.1) and (3.2.5) in [49].

With the notation $\mathrm{M}_G = \Lambda_F \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG)$ introduced in [49, (4.2.3)], the acyclic construction (4.2.6) or (4.3.7) for RG given in [49] according as Q is or is not free abelian looks like

$$\mathrm{M}(G)=\Gamma_L \otimes_\vartheta \mathrm{M}_G = \Gamma_L \otimes_\vartheta (\Lambda_F \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG)),$$

and its underlying chain complex is a free resolution of R in the category of right RG-modules. Here $\mathrm{M}(N) \otimes_{RN} RG$ is equipped with the tensor product algebra structure. Moreover, ξ and ϑ are the *twisting cochains* (4.2.1) and (4.3.5) in [49]. They are morphisms

$$\xi : \Lambda_F \to \mathrm{M}(N) \otimes_{RN} RG, \qquad \vartheta : \Gamma_L \to \mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\mathrm{M}_G),$$

and the property of being a twisting cochain means that, with the standard notation D for the differential in a Hom-complex,

$$D\xi = \xi \cup \xi \tag{5.2}$$

and

$$D\vartheta = \vartheta \cup \vartheta. \tag{5.3}$$

Furthermore, these twisting cochains are required to satisfy

$$(\varepsilon \otimes R\pi)\xi = \tau_F \colon A_F \to RF \tag{5.4}$$

and

$$g_e \vartheta = \iota t_Q \colon \Gamma_L \to \mathrm{end}_{RQ}(A_F \otimes_{\tau_F} RQ); \tag{5.5}$$

here $\mathrm{end}_{-}(-)$ refers to the differential graded subalgebra of the full endomorphism algebra $\mathrm{End}_{-}(-)$ which has no non-zero elements in negative degrees and which in degree zero consists of the equivariant chain maps only, and ι and g_e are the obvious morphisms

$$\iota \colon A_F \otimes_{\tau_F} RQ \to \mathrm{end}_{RQ}(A_F \otimes_{\tau_F} RQ),$$
$$g_e \colon \mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\mathrm{M}_G) \to \mathrm{end}_{RQ}(A_F \otimes_{\tau_F} RQ)$$

of augmented differential graded algebras. Notice that the object $\mathrm{M}(G)$ may be viewed as a contractible construction for the augmented differential graded algebra $\mathrm{M}(N) \otimes_{RN} RG$ with base $\bar{\mathrm{M}}(Q) = \mathrm{M}(Q) \otimes_{RQ} R$; cf. [49, (4.2.7) and (4.3.11)]. It is clear that the latter computes the homology of Q.

The reader will note that in order to see that the chain complex which underlies $\mathrm{M}(G)$ is a free resolution of R, it suffices to verify that ξ and ϑ are twisting cochains having the properties (5.4) and (5.5) as appropriate, no matter how these twisting cochains have been obtained. On the other hand, in [49] we used our machinery to *exhibit solutions* ξ and ϑ of (5.2) and (5.3), viewed as *equations in the unknowns* ξ and ϑ, respectively, subject to the constraints (5.4) or (5.5) as appropriate. One may view (5.2) and (5.3) as differential equations and (5.4) and (5.5) as boundary conditions.

6. Free Nilpotent Groups of Class 2

Let G be a free nilpotent group of class 2 on a certain set X. It fits into a central extension

$$\mathbf{e} \colon 0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} F \longrightarrow 1, \tag{6.1}$$

where $F = G^{\mathrm{Ab}}$ is free abelian of rank $|X|$, and where π is the obvious projection map. For convenience, we choose a linear ordering "<" for X; as usual, when the rank of F is infinite this may require some set theory. Now we can describe N as the free abelian group on a set Y consisting of the commutators

$$y = y(x_2, x_1) = [x_2, x_1] = x_2 x_1 x_2^{-1} x_1^{-1} \in G, \qquad x_1 < x_2,\ x_1 \in X,\ x_2 \in X.$$

Notice that N may be identified with the Schur multiplicator $\mathrm{H}_2(F)$ of F, and the extension (6.1) is sort of a universal central extension for F. In the present section we give an explicit description of our acyclic construction [49, (4.2.3')] for RG. This will serve as a universal example.

To apply the construction [49, (4.2.3')] to the extension $\mathbf{e}$, we need some preparations: To construct the small model [49, (3.1.1)] for N we introduce a linear ordering of the basis Y of N by means of the rule

$$y(x_2, x_1) < y(x_4, x_3) \qquad \text{if} \quad x_2 < x_4 \quad \text{or if} \quad x_2 \geqslant x_4 \quad \text{and} \quad x_1 \leqslant x_3,$$

so that, for $x_1 < x_2 < \cdots < x_n$,

$$\begin{aligned} y(x_2, x_1) < y(x_3, x_1) < \cdots < y(x_n, x_1) < \cdots < y(x_3, x_1) \\ < y(x_4, x_2) < \cdots < y(x_n, x_{n-1}). \end{aligned}$$

Let $W = \{w(x_2, x_1), x_1 < x_2 \in X\}$ be the set Y, regraded by 1—as usual we may identify W with the suspension of the set $\{y - 1, y \in Y\} \subseteq RN$, but we shall not need this—and let

$$\mathrm{M}(N) = \Lambda[W] \otimes_{\tau_N} RN, \quad \text{where } \tau_N(w(x_2, x_1)) = 1 - y(x_2, x_1),\ x_1 < x_2 \in X,$$

be the corresponding object (3.1.1) of [49]. This is just the well-known Koszul resolution of R over RN. Furthermore, let s be the contracting homotopy [49, (3.1.5)] of $\mathrm{M}(N)$, for $x \in X$, let $z(x) = \pi(x)$ so that $Z = \{z(x), x \in X\}$ is a basis for F as a free abelian group, and let $\sigma: F \to G$ be the section for π of the underlying sets given by

$$\sigma((z(x_1))^l (z(x_2))^m \cdot \ldots \cdot (z(x_k))^n) = x_1^l x_2^m \cdot \ldots \cdot x_k^n, \qquad x_1 < x_2 < \cdots < x_k.$$

Finally, let $\mathrm{M}(F) = \Lambda_F \otimes_{\tau_F} RF$ be the Koszul resolution of R in the category of RF-modules, written out as a bundle as in (3.1.1) of [49]. We then have the requisite data in place so that (4.1) and (4.2) in [49] apply: Let h and ∇ be the morphisms (4.1.2b) and (4.1.2a) respectively in [49] constructed from s and σ—they are given by

$$\nabla(q) = 1 \otimes \sigma(q), \qquad \text{for} \quad q \in Q, \tag{6.1.1}$$

$$h(a \otimes g) = s(ag(\sigma\pi g)^{-1}) \otimes (\sigma\pi g), \qquad \text{for} \quad a \in \mathrm{M}(N),\ g \in G, \tag{6.1.2}$$

and let $\xi: \Lambda_F \to \mathrm{M}(N) \otimes_{RN} RG$ be the twisting cochain in [49, (4.2.1)]; we remind the reader that it satisfies (5.4). The resulting acyclic construction

$\mathrm{M}(G) = \Lambda_F \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG)$ for RG of the kind [49, (4.2.3′)] is then explicit. We indicate this in low degrees:

PROPOSITION 6.2. *Write ξ in the form $\xi = \xi_1 + \xi_2 + \xi_3 + \cdots$ explained in* (2.11) *of* [49], *so that, for $p \geqslant 1$, ξ_p may be non-zero only on elements of degree p. Then the following hold, where $x_1 < x_2 < x_3 < x_4 \in X$, and where for $1 \leqslant i, j \leqslant 4$ the notation $v_i = v_{x_i}$, $y_{j,i} = y(x_j, x_i)$, $w_{j,i} = w(x_j, x_i)$ is used:*

$$\xi_1(v_x) = 1 - x, \qquad x \in X,$$

$$\xi_2(v_1 v_2) = w_{2,1} x_1 x_2,$$

$$\xi_3(v_1 v_2 v_3) = -(w_{3,1} w_{3,2} + w_{2,1} w_{3,1} y_{3,1} + w_{2,1} w_{3,1}) x_1 x_2 x_3,$$

$$\xi_4(v_1 v_2 v_3 v_4) = \sum x_1 x_2 x_3 x_4,$$

where $\sum$ is the sum of the following 16 *terms:*

$$\begin{array}{ll}
w_{2,1} w_{4,2} w_{4,3} y_{3,1} y_{4,1}, & w_{3,1} w_{4,2} w_{4,3} y_{4,1}, \\
w_{3,1} w_{3,2} w_{4,3} y_{4,1} y_{4,2}, & w_{4,1} w_{3,2} w_{4,3} y_{4,2}, \\
w_{4,1} w_{3,2} w_{4,2}, & w_{2,1} w_{4,1} w_{4,3}, \\
-w_{3,1} w_{4,1} w_{4,2} y_{3,2}, & w_{2,1} w_{3,1} w_{4,1} y_{4,1} y_{4,3}, \\
w_{4,1} w_{4,2} w_{4,3}, & w_{2,1} w_{3,2} w_{4,3} y_{3,1} y_{4,1} y_{4,2}, \\
w_{2,1} w_{3,2} w_{4,2} y_{3,1} y_{4,1}, & w_{3,1} w_{3,2} w_{4,2} y_{4,1}, \\
w_{2,1} w_{3,1} w_{4,3} y_{4,1}, & w_{2,1} w_{3,1} w_{4,1}, \\
-w_{2,1} w_{4,1} w_{3,2} y_{3,1}, & -w_{3,1} w_{4,2} w_{3,2}.
\end{array}$$

The proof is a tedious but straightforward evaluation of the formula given in (2.11) of [49]. We do not give the details here, but we mention that as p increases, the requisite calculations get more and more involved. However, when G is finitely generated, the computation of ξ is manifestly a finite problem. We spelled out the above 16 terms to indicate that the small free resolution is quite explicit. Also we mention that for a finitely generated free nilpotent group G of class 2 the twisting cochain ξ is of geometric significance, since it carries all the information that is needed to construct an aspherical manifold as the total space of a torus bundle over a torus having G as fundamental group by means of, e.g., a handle decomposition.

Remark 6.3. Let L be a Lie algebra, and let ΣL be its suspension; for our purposes this is just L, except its elements are regraded by 1. Then, as is well known, $\Sigma L \oplus L$ may be given the structure of a differential graded Lie algebra: For $x \in L$ write v_x for its suspension, and, for $x, y \in L$, let

$$[v_x, y] = [x, v_y] = v_{[x, y]}, \qquad [v_x, v_y] = 0, \qquad d(v_x) = x.$$

Then the resulting differential graded Lie algebra $\Sigma L \oplus_d L$ is obviously contractible.

Suppose now that the underlying R-module of L is free. Then L and $\Sigma L \oplus_d L$ admit universal enveloping algebras UL and $U(\Sigma L \oplus_d L)$. Moreover, the latter is in fact a contractible augmented differential graded algebra, and its underlying chain complex is the Koszul resolution of R over UL; cf., e.g., Exercise XIII.14 on page 287 of Cartan and Eilenberg [7]. The algebra $U(\Sigma L \oplus_d L)$ may also be written as a *Massey–Peterson* algebra $\Lambda(\Sigma L) \odot UL$; cf. Section 5 of May [53]. This construction works basically because in $\Sigma L \oplus_d L$ the Jacobi identity holds, which—as is well known—is equivalent to the associativity of the universal enveloping algebra. The Jacobi identity, in turn, implies the identity

$$[[x, v_y], z] + [x,[z, v_y]] + [[z, v_x], y] = 0, \qquad x, y, z \in L. \tag{6.4}$$

It is tempting to carry out a similar construction for a free nilpotent group G of class 2 of the kind considered above: For x, a free generator of G, and v, the suspension of a free generator, define a bracket $[x, v] = h[x, dv] \in \mathrm{M}(N) \otimes_{RN} RG$, where as usual $[x, dv] = x\,dv - (dv)x \in RG$; here h denotes the already mentioned contracting homotopy [49, (4.1.2b)] on $\mathrm{M}(N) \otimes_{RN} RG$ constructed from s and σ. This bracket then produces the term ξ_2 of the above twisting cochain. In fact, let $x_1 < x_2$ and, as above, write $v_1 = v_{x_1}$, etc.; we may then formally compute

$$\begin{aligned} d(v_1 v_2) &= (dv_1)v_2 - v_1(dv_2) = (x_1 - 1)v_2 - v_1(x_2 - 1) \\ &= [x_1, v_2] + v_2(x_1 - 1) - v_1(x_2 - 1) \\ &= -\xi(v_1 v_2) + v_2(x_1 - 1) - v_1(x_2 - 1), \end{aligned}$$

since by construction $\xi_2(v_1 v_2) = -h[x_1, x_2]$. However, when G has more than two free generators, this bracket will *not* satisfy the Jacobi identity. In fact, instead of the formula (6.4) we now have

$$[[x_2, v_3], x_1] + [x_2,[x_1, v_3]] + [[x_1, v_2], x_3] = d(\xi_3(v_1 v_2 v_3)).$$

When G has more than three free generators, even higher terms necessarily come into play. This is related with the deformation theory of Schlessinger and Stasheff [40] for Lie algebras and, more generally, what are called s(trong) h(omotopy) Lie algebras.

On the other hand, the construction $\mathrm{M}(G) = \Lambda_F \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG)$ may be viewed as a certain non-associative differential graded algebra with higher homotopies. This is related with the perturbation theory in Gugenheim and Stasheff [47]. For our purposes the coalgebra structure of Λ_F provides the requisite combinatorics which organises the way in which the differential is written as a derivation with respect to the non-associative algebra structure.

At this stage it would be easy to compute a number of cohomology rings of free nilpotent groups of class two with few generators, but it is not clear whether such calculations would provide any insight now. Explicit calculations of appropriate examples will be carried out in later sections. However, it seems worthwhile mentioning that by means of computer calculations Lambe [50] exhibited torsion in the integral cohomology of a free nilpotent group of class two on at least four generators.

7. Nilpotent Groups of Class Two with Free Abelian Quotient

7.1. Let N be an arbitrary abelian and F a free abelian group, and let G be a central extension

$$0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} F \longrightarrow 1 \tag{7.1.1}$$

as indicated. In the present section we apply our approach to groups of the kind G. We shall use the material given in the previous section.

Let Z be a basis for F, let $\tilde{G}$ be the free nilpotent group of class two on Z, let $\tilde{G} \to F$ be the obvious surjective homomorphism, and lift the identity map of F to a homomorphism $\tilde{G} \to G$, so that there results a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{N} \arrow[r] \arrow[d, "\phi"'] & \tilde{G} \arrow[r] \arrow[d, "\psi"'] & F \arrow[r] \arrow[d, "\mathrm{Id}"'] & 1 \\ 0 \arrow[r] & N \arrow[r] & G \arrow[r] & F \arrow[r] & 1; \end{tikzcd} \tag{7.1.2}$$

its top row is an extension of the kind (6.1.1). Further, let $\mathrm{M}(N)$ and $\mathrm{M}(\tilde{N})$ be objects of the kind [49, (3.1.2)] and [49, (3.2.6)], respectively, let $\tilde{\xi}\colon \Lambda_F \to \mathrm{M}(\tilde{N}) \otimes_{R\tilde{N}} R\tilde{G}$ be the twisting cochain in (6.2) above (and denoted by ξ there), let $\phi_\flat\colon \mathrm{M}(\tilde{N}) \to \mathrm{M}(N)$ be a morphism of augmented differential graded algebras which is a comparison map for ϕ, and let

$$\xi = (\phi_\flat \otimes R\psi)\, \tilde{\xi}\colon \Lambda_F \to \mathrm{M}(N) \otimes_{RN} RG; \tag{7.1.3}$$

an explicit construction for $\phi_\flat$ has been given in (3.3.3) of [49]. Then it is manifest that ξ is a twisting cochain satisfying (5.4), i.e., it is one of the kind [49, (4.2.1)]. As explained in (4.2.5) of [49] and in Section 5, the bundle

$$\mathrm{M}(G) = \Lambda_F \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG),$$

which is of the kind [49, 4.2.3′)], is an acyclic construction for RG, and hence its underlying chain complex is a small free resolution for G.

We now employ our construction to calculate a number of examples. We shall also illustrate the diagonal map for $\mathrm{M}(G)$ given in (4.5.3a) of [49]. We give these examples in order to demonstrate that our approach really works. It is clear that more subtle examples than the ones we are about to present can be examined by means of our method.

7.2. Consider the group

$$G = \langle x_1, x_2, y; [x_1, y] = 1; [x_2, y] = 1, [x_2, x_1] = y^k \rangle. \tag{7.2.1}$$

It is sometimes referred to as the *generalized Heisenberg group*, since for $k = 1$ it is just the ordinary Heisenberg group. The subgroup N generated by y is central, and our machinery applies to the corresponding group extension. Now the group $\tilde{G}$ in the diagram (7.1.2) has the presentation

$$\tilde{G} = \langle x_1, x_2, \tilde{y}; [x_1, \tilde{y}] = 1; [x_2, \tilde{y}] = 1, [x_2, x_1] = \tilde{y} \rangle,$$

the groups N and $\tilde{N}$ are both free cyclic, generated by y and $\tilde{y}$, respectively, and ϕ and ψ are the obvious maps; in particular, ϕ sends $\tilde{y}$ to y^k. From the previous section we know that, with the notation adjusted suitably, the corresponding unreduced bundle of the kind [49, (4.2.3′)] for the top row of (7.1.2) looks like

$$\mathrm{M}(\tilde{G}) = \Lambda[v_1, v_2] \otimes_{\tilde{\xi}} (\Lambda[v_{\tilde{y}}] \otimes_{\tau_{\tilde{N}}} R\tilde{G}),$$

where $\tau_{\tilde{N}}(v_{\tilde{y}}) = 1 - \tilde{y}$, $\tilde{\xi}(v_1) = 1 - x_1$, $\tilde{\xi}(v_2) = 1 - x_2$, $\tilde{\xi}(v_1 v_2) = v_{\tilde{y}} x_1 x_2$. Moreover, a comparison map

$$\phi_\flat : \mathrm{M}(\tilde{N}) = \Lambda[v_{\tilde{y}}] \otimes_{\tau_{\tilde{N}}} R\tilde{N} \to \Lambda[v_y] \otimes_{\tau_N} RN = \mathrm{M}(N)$$

of the desired kind is given by the formula $\phi_\flat(v_{\tilde{y}}) = v_y(1 + y + \cdots + y^{k-1})$. Hence the resulting object for G looks like

$$\mathrm{M}(G) = \Lambda[v_1, v_2] \otimes_\xi (\Lambda[v_y] \otimes_{\tau_N} RG),$$

where $\tau_N(v_y) = 1 - y$, and where

$$\xi(v_1) = 1 - x_1,\ \xi(v_2) = 1 - x_2,\ \xi(v_1 v_2) = v_y(1 + y + \cdots + y^{k-1})\, x_1 x_2.$$

Next, to get a diagonal map, we recall at first from (4.5) of [49] that its construction can be carried out compatibly with the diagonal map Δ_N on $\mathrm{M}(N)$ given as (3.4.3b) of [49]. Now, after the appropriate parameters have been fixed, (4.5.3a) of [49] yields the following formula for an $\mathrm{M}(N)\otimes_{RN} RG$-equivariant diagonal map Δ_G for $\mathrm{M}(G)$:

$$\Delta_G(v_i) = v_i \otimes 1 + 1 \otimes v_i, \qquad i = 1, 2,$$

$$\Delta_G(v_1v_2) = \begin{cases} v_1v_2 \otimes x_1x_2 - v_2 \otimes v_1x_2 + 1 \otimes v_1v_2 \\ +v_1 \otimes (v_2x_1 - v_y(1 + y + \cdots + y^{k-1})\, x_1x_2) \\ -(v_2x_1 - v_y\, y_k x_1x_2) \otimes v_y\, y_k x_1x_2 \\ -\sum\limits_{0<\beta<k} (v_y(1 + y + \cdots + y^{\beta-1})\, x_1x_2) \otimes v_y\, y^\beta x_1x_2; \end{cases} \tag{7.2.2}$$

here we have written $y_k = 1 + y + \cdots + y^{k-1}$ for short, and "equivariant" must be understood with respect to the tensor product diagonal map

$$\Delta : \mathrm{M}(N)\otimes_{RN} RG \to (\mathrm{M}(N)\otimes_{RN} RG) \otimes (\mathrm{M}(N)\otimes_{RN} RG)$$

which turns $\mathrm{M}(N)\otimes_{RN} RG$ into an augmented differential graded algebra with diagonal; cf. (4.1.2) of [49]. We do not reproduce the requisite calculations.

It follows that, with a minor abuse of the notation ξ, the corresponding reduced object [49, (4.2.6)] may be written

$$\bar{\mathrm{M}}(G) = \Lambda[v_1, v_2] \otimes_\xi \Lambda[v_y],$$

where $\xi(v_1v_2) = kv_y$, whence the only non-zero differential is given by $d(v_1v_2) = -kv_y$. Actually, this abuse of notation is justified by the observation that the obvious morphism

$$\mathrm{M}(N)\otimes_{RN} RG \to \bar{\mathrm{M}}(N) = \Lambda[v_y]$$

of augmented differential graded algebras induces the structure of a differential graded $\mathrm{M}(N)\otimes_{RN} RG$-module on $\Lambda[v_y]$. Moreover, (7.2.2) entails at once the formulas (7.2.3) below for the induced diagonal map (say) $\Delta_G : \bar{\mathrm{M}}(G) \to \bar{\mathrm{M}}(G) \otimes \bar{\mathrm{M}}(G)$ on $\bar{\mathrm{M}}(G)$:

$$\Delta_G(v_i) = v_i \otimes 1 + 1 \otimes v_i, \qquad i = 1, 2,$$

$$\Delta_G(v_1v_2) = \begin{cases} v_1v_2 \otimes 1 - v_2 \otimes v_1 + 1 \otimes v_1v_2 \\ +v_1 \otimes (v_2 - kv_y) \\ -(v_2 - kv_y) \otimes kv_y \\ -\dfrac{k(k-1)}{2} v_y \otimes v_y, \end{cases} \tag{7.2.3}$$

$$\Delta_G(v_1v_2v_y) = \begin{cases} v_1v_2v_y \otimes 1 - v_1v_y \otimes v_2 + v_1 \otimes v_2v_y \\ +v_1v_2 \otimes v_y + kv_1v_y \otimes v_y. \end{cases}$$

It is straightforward to compute the cohomology ring $\mathrm{H}^*(G)=\mathrm{H}^*(G, R)$ from this. To write it down, we denote by ω_1, ω_2, and ω_y the duals of respectively v_1, v_2, and v_y in the basis of monomials, and we do not distinguish in notation between cocycles and the classes they represent.

For $R=\mathbf{Z}$,

$$\begin{aligned} &\mathrm{H}^1(G)=\mathbf{Z}^2, && \text{generated by } \omega_1, \omega_2,\\ &\mathrm{H}^2(G)=\mathbf{Z}^2\oplus\mathbf{Z}/k, && \text{generated by } \omega_1\omega_y, \omega_2\omega_y, \omega_1\omega_2,\\ &\mathrm{H}^3(G)=\mathbf{Z}, && \text{generated by } \omega_1\omega_2\omega_y, \end{aligned}$$

and the multiplication rule is the obvious one. In particular, $\mathrm{H}^*(G, \mathbf{Z})$ is additively and multiplicatively isomorphic to the associated graded object coming from the corresponding group extension (7.1.1).

Next let R be a ring whose characteristic divides the number k. Then, as a graded commutative algebra, $\mathrm{H}^*(G)=\mathrm{H}^*(G, R)$ is generated by ω_1, ω_2, ω_y, all of degree 1, subject to the relations

$$\omega_1^2=\omega_2^2=[\omega_1, \omega_2]=[\omega_1, \omega_y]=[\omega_2, \omega_y]=0, \qquad \omega_y^2=\frac{k(k-1)}{2}\,\omega_1\omega_2.$$

Thus when k is odd, $\omega_y^2=0$ (as it should be for formal reasons anyway), while when k is even, $\omega_y^2=(k/2)\,\omega_1\omega_2$, and this is zero only if the characteristic of R divides $k/2$. Thus we see that $\mathrm{H}^*(G)$ is still additively isomorphic to the corresponding associated graded object and that for k odd this is also multiplicatively true, while for k even it is true only if the characteristic of R divides $k/2$.

Remark 7.2.4. Write $\bar{\mathrm{M}}(\tilde{G})=\Lambda[v_1, v_2]\otimes_{\tilde{\xi}}\Lambda[v_{\tilde{y}}]$ for the corresponding reduced object for $\tilde{G}$ of the kind [49, (4.2.6)]. In view of what was said above, the corresponding reduced object $\bar{\mathrm{M}}(G)=\Lambda[v_1, v_2]\otimes_{\xi}\Lambda[v_y]$ for G of the kind [49, (4.2.6)] may be viewed as being induced from $\bar{\mathrm{M}}(\tilde{G})$ by means of the algebra map $\Lambda[v_{\tilde{y}}]\to\Lambda[v_y]$ which sends $v_{\tilde{y}}$ to kv_y. The above term $(k(k-1)/2)\,v_y\otimes v_y$ in the diagonal map for $\bar{\mathrm{M}}(G)$ indicates that the diagonal map for $\bar{\mathrm{M}}(G)$ cannot in general be induced from the one for $\bar{\mathrm{M}}(\tilde{G})$: we have seen that when k is even and when the characteristic of R does not divide $k/2$, the dual of this term yields a non-trivial contribution to the ring structure of $\mathrm{H}^*(G)$.

7.3. Let

$$G=\langle x_1, x_2, y;\ [x_1, y]=1,\ [x_2, y]=1,\ [x_2, x_1]=y^k,\ y^l=1,\ 0\leqslant k\leqslant l\rangle. \tag{7.3.1}$$

We could in fact simplify this description somewhat by assuming that k divides l, but this would not simplify the exposition below. To apply our method, let N be the central subgroup of G generated by y; it is cyclic of order l. Further, to relate the present example with the one dealt with in the previous subsection, we now denote the groups written N and G in the previous subsection by $\tilde{N}$ and $\tilde{G}$, respectively. It is obvious that the group G fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{G} & \longrightarrow & F & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \end{array} \tag{7.3.2}$$

where the top row is the group extension considered in the previous subsection and where ϕ and ψ are the obvious projection maps. Let

$$\tilde{G} = \langle x_1, x_2, \tilde{y}; [x_1, \tilde{y}] = 1, [x_2, \tilde{y}] = 1, [x_2, x_1] = \tilde{y}^k \rangle,$$

and let

$$\mathrm{M}(\tilde{G}) = \Lambda[v_1, v_2] \otimes_{\tilde{\xi}} (\Lambda[v_{\tilde{y}}] \otimes_{\tau_{\tilde{N}}} R\tilde{G})$$

be the unreduced object for $\tilde{G}$ of the kind [49, (4.2.3′)] given in the previous subsection, with the notation suitably adjusted; further, let $\mathrm{M}(N) = \Gamma[u_y] \otimes_{t_N} (\Lambda[v_y] \otimes_{\tau_N} RN)$ be the standard small free resolution for the cyclic group $N = \langle y; y^l = 1 \rangle$ of order l, written out as in [49, (3.2.14)] as indicated so that

$$\tau_N(v_y) = 1 - y, \qquad t_N(u_y) = -v_y(1 + y + \cdots + y^{l-1}),$$

and let

$$\phi_\flat : \mathrm{M}(\tilde{N}) = \Lambda[v_{\tilde{y}}] \otimes_{\tau_{\tilde{N}}} R\tilde{N} \to \mathrm{M}(N) = \Gamma[u_y] \otimes_{t_N} (\Lambda(v_y] \otimes_{\tau_N} RN)$$

be the obvious comparison map for the surjection $\tilde{N} \to N$; it is given by $\phi_\flat(v_{\tilde{y}}) = v_y$, $\phi_\flat(\tilde{y}) = y$, and it is manifestly a morphism of augmented differential graded algebras with diagonal. A corresponding unreduced bundle $\mathrm{M}(G)$ for G of the kind [49, (4.2.3′)] therefore looks like

$$\mathrm{M}(G) = \Lambda[v_1, v_2] \otimes_{\xi} (\Gamma[u_y] \otimes_{t_N} (\Lambda[v_y] \otimes_{\tau_N} RG)),$$

where

$$\begin{aligned} \tau_N(v_y) &= 1 - y \in RN \subseteq RG, \\ t_N(u_y) &= -v_y(1 + y + \cdots + y^{l-1}) \in \Lambda[v_y] \otimes RN \subseteq \Lambda[v_y] \otimes RG, \\ \xi(v_1) &= 1 - x_1, \\ \xi(v_2) &= 1 - x_2, \\ \xi(v_1 v_2) &= v_y(1 + y + \cdots + y^{k-1})\, x_1 x_2. \end{aligned}$$

Moreover, a diagonal map Δ_G for $\mathrm{M}(G)$ is the $(\mathrm{M}(G)\otimes_{RN} RG)$-equivariant map given on v_1, v_2, and v_1v_2 by formally the same formulas as those given in (7.2.2) above.

The corresponding reduced object [49, (4.2.6)] looks like

$$\bar{\mathrm{M}}(G)=(\Lambda[v_1, v_2]\otimes\Gamma[u_y]\otimes\Lambda[v_y], d)$$

and the differential d is given by the formulas

$$\begin{aligned}
d(\gamma_i(u_y)) &= l(\gamma_{i-1}(u_y))\,v_y, && i\geqslant 1,\\
d((\gamma_i(u_y))\,v_y) &= 0, && i\geqslant 0,\\
d(v_\nu(\gamma_i(u_y))) &= -lv_\nu(\gamma_{i-1}(u_y))\,v_y, && i\geqslant 1,\ \nu=1, 2,\\
d(v_\nu(\gamma_i(u_y))\,v_y) &= 0, && i\geqslant 0,\ \nu=1, 2,\\
d(v_1v_2) &= -kv_y,\\
d(v_1v_2(\gamma_i(u_y))) &= -k(\gamma_i(u_y))\,v_y+lv_1v_2(\gamma_{i-1}(u_y))\,v_y, && i\geqslant 1,\\
d(v_1v_2(\gamma_i(u_y))\,v_y) &= 0, && i\geqslant 1.
\end{aligned}$$

Furthermore, a diagonal map (say) Δ_G for $\bar{\mathrm{M}}(G)$ is the $\bar{\mathrm{M}}(N)=\Gamma[u_y]\otimes\Lambda[v_y]$-equivariant map given on v_1, v_2, and v_1v_2 by formally the same formulas as those spelt out in (7.2.3).

It is straightforward to compute the cohomology ring $\mathrm{H}^*(G)=\mathrm{H}^*(G, R)$ from this: We denote by ω_1, ω_2, ω_y, c_y the duals of respectively v_1, v_2, v_y, u_y in the basis of monomials. Then the dual object $\bar{\mathrm{M}}(G)^*$ looks like

$$\bar{\mathrm{M}}(G)^*=(\Lambda[\omega_1, \omega_2]\otimes\mathrm{P}[c_y]\otimes\Lambda[\omega_y], d),$$

where P refers to the polynomial algebra on the indicated generator, and the differential d is given by the formulas

$$\begin{aligned}
d(c_y^i) &= 0, && i\geqslant 1,\\
d(\omega_1c_y^i) &= 0, && i\geqslant 0,\\
d(\omega_2c_y^i) &= 0, && i\geqslant 0,\\
d(\omega_1\omega_2c_y^i) &= 0, && i\geqslant 0,\\
d(c_y^i\omega_y) &= k\omega_1\omega_2c_y^i+lc_y^{i+1}, && i\geqslant 0,\\
d(\omega_1c_y^i\omega_y) &= l\omega_1c_y^{i+1}, && i\geqslant 0,\\
d(\omega_2c_y^i\omega_y) &= l\omega_2c_y^{i+1}, && i\geqslant 0,\\
d(\omega_1\omega_2c_y^i\omega_y) &= l\omega_1\omega_2c_y^{i+1}, && i\geqslant 0.
\end{aligned}$$

Here and henceforth we systematically use the Eilenberg–Koszul convention to the effect that the coboundary dc of a cochain c is given by $d(c)=$

$(-1)^{|c|+1} cd$ and that, for example, $(\omega_1\omega_2)(v_1 v_2) = -1$. Notice the above differential is a derivation with respect to the induced $\Lambda[\omega_1, \omega_2]$-module structure on $\bar{\mathrm{M}}(G)^*$ but it is *not* a derivation with respect to the tensor product algebra structure on the latter. To write down formulas for the cohomology of G, as before, we do not distinguish in notation between cocycles and cohomology classes represented by them; further, we write $(\, , \,)$ for the greatest common divisor.

For $R = \mathbf{Z}$ the cohomology ring $\mathrm{H}^*(G)$ is the graded commutative algebra generated by ω_1, ω_2, c_y subject to the relations

$$l\omega_i c_y^j = 0, \qquad l\omega_1\omega_2 c_y^j = 0, \qquad -k\omega_1\omega_2 c_y^{j-1} = lc_y^j, \qquad \omega_i^2 = 0,$$

where $i = 1, 2$, and $j \geqslant 1$. Hence

$$\begin{aligned} \mathrm{H}^1(G) &= \mathbf{Z} \oplus \mathbf{Z}, && \text{generated by } \omega_1 \text{ and } \omega_2, \\ \mathrm{H}^2(G) &= \mathbf{Z} \oplus \mathbf{Z}/(l, k), && \text{generated by } c_y \text{ and } \omega_1\omega_2, \end{aligned}$$

and, for $j > 1$,

$$\begin{aligned} \mathrm{H}^{2j-1}(G) &= \mathbf{Z}/l \oplus \mathbf{Z}/l, && \text{generated by } \omega_1 c_y^{j-1} \text{ and } \omega_2 c_y^{j-1}, \\ \mathrm{H}^{2j}(G) &= \left(\mathbf{Z}\Big/\frac{l^2}{(l,k)}\right) \oplus \mathbf{Z}/(l, k), && \text{generated by } c_y^j \text{ and } \omega_1\omega_2 c_y^{j-1}. \end{aligned}$$

Notice that for $j > 1$ the indicated generators do *not* generate the indicated direct summands. It is interesting to observe that when $(l, k) < l$, the group G has exponent $l^2/(l, k)$ occurring infinitely often in $H^*(G)$ but each finite subgroup of G has order strictly smaller than $l^2/(l, k)$.

It is instructive to point out that $\mathrm{H}^*(G)$ is in general not even additively isomorphic to the associated graded object coming from the corresponding group extension (7.1.1): The cohomology spectral sequence has

$$E_2 = \mathrm{H}^*(F, \mathrm{H}^*(N)) = (\Lambda[\omega_1, \omega_2] \otimes \mathrm{P}[c_y])/lc_y,$$

and it is not hard to see that $E_2 = E_\infty$. When $(l, k) = 1$, for $j > 1$ the group $\mathrm{H}^{2j}(G)$ equals $\mathbf{Z}/l^2$, while its associated graded group equals

$$E_2^{0,2j} \oplus E_2^{2,2j-2} = \mathbf{Z}/l \oplus \mathbf{Z}/l.$$

When there is torsion in R, various special cases come up. Suppose, for example, that l and k are even, and suppose that the characteristic of R divides l and k. Then, as a graded commutative algebra, $\mathrm{H}^*(G)$ is generated by $\omega_1, \omega_2, c_y, \omega_y$ subject to the relations

$$\omega_y^2 = \frac{k}{2}\omega_1\omega_2 - \frac{l}{2}c_y, \qquad \omega_1^2 = 0, \qquad \omega_2^2 = 0.$$

Thus when $k=2$, $l=4$, $R=\mathbf{Z}/2$, we get the relation $\omega_y^2=\omega_1\omega_2$. Likewise, let $k=1$, let p be a prime which divides l, and let $R=\mathbf{Z}/p$. Then $\tilde{G}$ is the Heisenberg group G_H, and inspection shows that, as a chain complex, $\bar{\mathrm{M}}(G)\cong\Gamma[u_y]\otimes\bar{\mathrm{M}}(G_H)$. Hence $\mathrm{H}^*(G)\cong\mathrm{P}[c_y]\otimes\mathrm{H}^*(G_H)$, and this is also multiplicatively true. This is clear if p is odd, while if $p=2$ we have

$$(\omega_1\omega_y)(\omega_2\omega_y)=\frac{l}{2}\,\omega_1\omega_2c_y=\frac{l^2}{2}\,c_y^2=0.$$

8. General Nilpotent Groups of Class 2

8.1. *The Small Free Resolution*

Let N and Q be arbitrary abelian groups, and let

$$\mathbf{e}: 0 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1 \tag{8.1.1}$$

be a central extension. We now apply the recipe in (4.3) of [49] to $\mathbf{e}$ to obtain a free resolution of R in the category of right RG-modules. We maintain the notation in (4.3) of [49], and we remind the reader that this notation has to some extent been reproduced in Section 5. The group G_F coming into play below has already been dealt with in (7.1) above; it was denoted there by G.

Let $\xi_F: A_F\to \mathrm{M}(N)\otimes_{RN} RG_F$ be the twisting cochain (7.1.3) given in the previous section and denoted by ξ there. Then the composite

$$\xi: A_F\to\mathrm{M}(N)\otimes_{RN} RG_F\to\mathrm{M}(N)\otimes_{RN} RG \tag{8.1.2}$$

of ξ_F with the obvious surjection (cf. (5.1)) as indicated is a twisting cochain which has the property (5.4). The *new* ingredient which, due to the non-freeness of Q, now comes into play is the corresponding twisting cochain

$$\vartheta: \Gamma_L\to \mathrm{end}_{\mathrm{M}(N)\otimes_{RN} RG}(\mathrm{M}_G),$$

given in (4.3.5) of [49]; cf. Section 5. It gives rise to the twisted tensor product $\mathrm{M}(G)=\Gamma_L\otimes_\vartheta \mathrm{M}_G$ introduced in (4.3.7), and in view of (4.3.8) the chain complex which underlies $\mathrm{M}(G)$ is a free resolution of R in the category of right RG-modules. The main labor in the present section is the explicit construction of ϑ.

8.2. *The Examples*

We shall illustrate our construction with a number of examples of groups $G=G(m_1, m_2, l)$ given by a presentation

$$G=\langle x_1, x_2, y; [x_2, x_1]=y, [x_2, y]=1,$$
$$[x_1, y]=1, x_1^{m_1}=1, x_2^{m_2}=1, y^l=1\rangle,$$

where m_1, m_2, and l are appropriate non-negative integers, and where $l > 1$. To avoid trivial cases we shall assume that $m_1 \neq 1 \neq m_2$, and we shall also assume that l divides m_1 and m_2, so that $m_1 = k_1 l$ and $m_2 = k_2 l$ for appropriate numbers k_1 and k_2, but we do not exclude the cases $m_1 = 0$ or $m_2 = 0$ (or both). Notice that the elements x_1^l and x_2^l lie in the centre of G.

The subgroup N generated by y is central in G, and we apply our machinery to the central extension

$$\mathbf{e}: 0 \to N \to G \to Q \to 1$$

of $N = \mathbf{Z}/l$ by $Q = (\mathbf{Z}/m_1) \times (\mathbf{Z}/m_2)$ (with the convention $(\mathbf{Z}/0) = \mathbf{Z}$). To this end we choose the obvious free presentation

$$0 \to L \to F \to Q \to 1$$

in the category of abelian groups, where $F = \mathbf{Z} \times \mathbf{Z}$, and where $L = \mathbf{Z} \times \mathbf{Z}$ or $L = \mathbf{Z}$ as appropriate, generated by $x_1^{m_1}$ and $x_2^{m_2}$. The pullback group G_F (cf. (5.1)) is a group of the kind examined in (7.3) above and denoted by G there. Further, an object of the kind [49, (3.2.14)] looks like

$$\mathrm{M}(N) = \Gamma[u_y] \otimes_{t_N} (\Lambda[v_y] \otimes_{\tau_N} RN),$$

where

$$\tau_N(v_y) = 1 - y \in RN, \qquad t_N(u_y) = -v_y(1 + y + \cdots + y^{l-1}).$$

Hence, with a slight abuse of the notation τ_N and t_N, the corresponding algebra $\mathrm{M}(N) \otimes_{RN} RG$ introduced in [49, (4.1)] looks like

$$\mathrm{M}(N) \otimes_{RN} RG = \Gamma[u_y] \otimes_{t_N} (\Lambda[v_y] \otimes_{\tau_N} RG).$$

Further, in view of (7.3) and (8.1), the corresponding bundle [49, (4.2.3)] looks like

$$\mathrm{M}_G = \Lambda[v_1, v_2] \otimes_\xi (\mathrm{M}(N) \otimes_{RN} RG),$$

where

$$\xi(v_1) = 1 - x_1, \qquad \xi(v_2) = 1 - x_2, \qquad \xi(v_1 v_2) = v_y x_1 x_2.$$

In order to describe the result of carrying out the construction of the corresponding twisting cochain $\vartheta: \Gamma_L \to \mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\mathrm{M}_G)$ of the kind [49, (4.3.5)], we need some preparations: We order the obvious basis of $\Lambda_F = \Lambda[v_1, v_2]$ by means of the rule $1 < v_1 < v_2 < v_1 v_2$. Since this basis is as well one of M_G viewed as a right $\mathrm{M}(N) \otimes_{RN} RG$-module, the elements Θ

of the underlying algebra of $\mathrm{End}_{\mathrm{M}(N)\otimes_{RN} RG}(\mathrm{M}_G)$ may be written as matrices

$$\Theta = [\Theta(1)\ \Theta(v_1)\ \Theta(v_2)\ \Theta(v_1v_2)]$$

of column vectors $\Theta(v)$ as indicated, and the elements of the subalgebra $\mathrm{end}_{\mathrm{M}(N)\otimes_{RN} RG}(\mathrm{M}_G)$ may clearly be written out in the same way. Since M_G is a right $\mathrm{M}(N)\otimes_{RN} RG$-module, composition of operators corresponds to the usual multiplication of matrices. We remind the reader that $\mathrm{end}_{-}(-)$ refers to the subalgebra of the full endomorphism algebra $\mathrm{End}_{-}(-)$ which has no non-zero elements in negative degrees and which in degree zero consists of the equivariant chain maps only. To describe the twisting cochain ϑ concisely, we introduce the following elements of RG:

$$\begin{aligned}
b_i &= 1 + x_i + \cdots + x_i^{l-1}, && i = 1, 2,\\
A_i &= 1 + x_i^l + x_i^{2l} + \cdots + x_i^{(k_i-1)l}, && i = 1, 2,\\
D_1 &= x_1x_2 + (1+y)\,x_1^2x_2 + \cdots + (1 + y + \cdots + y^{l-2})\,x_1^{l-1}x_2, &&\\
D_2 &= x_1x_2 + (1+y)\,x_1x_2^2 + \cdots + (1 + y + \cdots + y^{l-2})\,x_1x_2^{l-1}, &&\\
Y &= 1 + y + \cdots + y^{l-1}, &&
\end{aligned}$$

with the convention $A_i = b_i = 0$, if $m_i = 0$, where $i = 1, 2$. Notice that the elements A_1, A_2, Y lie in the centre of RG. Finally, we introduce the operators

$$U_1 = \begin{bmatrix} 0 & 0 & 0 & 0\\ -b_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -b_1 & 0 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -b_2 & 0 & 0 & 0\\ 0 & b_2 & 0 & 0 \end{bmatrix},$$

$$U_{1,k} = \begin{bmatrix} 0 & 0 & -(\gamma_k(u_y))x_2 & (\gamma_k(u_y))\,v_yx_1x_2\\ 0 & 0 & (\gamma_{k-1}(u_y))\,v_y\,D_1 & (\gamma_k(u_y))x_2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad k \geqslant 0,$$

$$U_{2,k} = (-1)^{k+1}\begin{bmatrix} 0 & (\gamma_k(u_y))x_1 & 0 & -(\gamma_k(u_y))\,v_yx_1x_2\\ 0 & 0 & 0 & 0\\ 0 & -(\gamma_{k-1}(u_y))\,v_y\,D_2 & 0 & (\gamma_k(u_y))x_1\\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad k \geqslant 0,$$

with the convention $\gamma_{-1}(u_y) = 0$. Notice that for $k \geqslant 1$ the operators $U_{1,k}$ and $U_{2,k}$ lie in $\mathrm{end}_{\mathrm{M}(N)\otimes_{RN} RG}(\mathrm{M}_G)$, but this is not true of $U_{1,0}$ and $U_{2,0}$, since these have negative degrees, while by definition, the elements of $\mathrm{end}_{\mathrm{M}(N)\otimes_{RN} RG}(\mathrm{M}_G)$ have non-negative degrees.

8.3. *The Case* $m_1 > 1$, $m_2 = 0$

Now $Q = (\mathbf{Z}/m_1) \times \mathbf{Z}$, and a tedious but straightforward application of the recipe for the twisting cochain $\vartheta\colon \Gamma[u_1] \to \operatorname{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\mathrm{M}_G)$ of the kind [49, (4.3.5)] yields the formulas

$$\vartheta(u_1) = A_1(U_1 + U_{1,1}), \qquad \vartheta(\gamma_k(u_1)) = A_1^k U_{1,k}, \qquad k \geqslant 2. \tag{8.3.1}$$

We supress the details of the calculation. However, we indicate a proof of the characterising property (5.3) in order for ϑ to be a twisting cochain: Since Γ_L has zero differential, the rule (5.3) amounts to

$$\begin{aligned} &D^\xi(\vartheta(u_1)) = 0, \\ &D^\xi(\vartheta(\gamma_{k+1}(u_1))) = \sum_{a+b=k+1} (\vartheta(\gamma_a(u_1)))(\vartheta(\gamma_b(u_1))), \qquad k \geqslant 1, \end{aligned} \tag{8.3.2}$$

where D^ξ is the *twisted* differential in $\operatorname{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\mathrm{M}_G)$, the twisting being induced from the twisting cochain ξ. Explicitly, for an operator Θ,

$$D^\xi(\Theta) = D(\Theta) - (\xi \cap\ )\Theta - (-1)^{|\Theta|}\, \Theta(\xi \cap\ ), \tag{8.3.3}$$

where D is the (untwisted) differential in

$$\operatorname{end}_{\mathrm{M}(N) \otimes_{RN} RG}(\Lambda_F \otimes (\mathrm{M}(N) \otimes_{RN} RG))$$

induced by the differential on $\mathrm{M}(N)$; for clarity we remind the reader that without the differentials $\mathrm{M}_G = \Lambda_F \otimes (\mathrm{M}(N) \otimes_{RN} RG)$. Now (8.3.3) yields

$$\begin{aligned} D^\xi(\vartheta(u_1)) &= A_1(D(U_1 + U_{1,1}) - (U_1 + U_{1,1})(\xi \cap\ ) - (\xi \cap\ )(U_1 + U_{1,1})) \\ &= A_1(D^\xi U_{1,1} - (U_1(\xi \cap\ ) + (\xi \cap\ )\, U_1)) \\ &= A_1(D^\xi U_{1,1} - (U_1 U_{1,0} + U_{1,0} U_1)) \end{aligned}$$

and a calculation yields

$$\sum_{a+b=k+1} (\vartheta(\gamma_a(u_1)))(\vartheta(\gamma_b(u_1))) = A_1^{k+1}(U_1 U_{1,k} + U_{1,k} U_1).$$

Hence to see that ϑ is a twisting cochain, it suffices to verify the formulas

$$A_1^{k+1}(D^\xi(U_{1,k+1}) - (U_1 U_{1,k} + U_{1,k} U_1)) = 0, \qquad k \geqslant 0.$$

These formulas, in turn, can easily be checked by means of (8.3.3) and the following commutation rules in RG:

$$\begin{aligned} (y-1)\, D_1 &= x_2 b_1 - b_1 x_2 \\ x_1 x_2 b_1 + (x_1 - 1)\, D_1 &= Y x_1^l x_2 \\ b_1 x_1 x_2 + D_1(x_1 - 1) &= Y x_1^l x_2. \end{aligned} \tag{8.3.4}$$

The proof of these commutation rules is left to the reader.

The reader will note that these commutation rules reflect the structure of G, and we thereby see how the structure of G gives rise to non-trivial higher terms in ϑ.

8.4. *The Integral Cohomology of* $G(m_1, 0, l)$

Now we have the machinery in place to compute the homology and cohomology of $G = G(m_1, 0, l)$ for arbitrary coefficients. For simplicity, we only give a calculation of the integral cohomology ring in number of special cases.

By construction, the reduced object $\bar{\mathrm{M}}(G)$ of the kind [49, (4.3.10)] may be written as a bundle $\bar{\mathrm{M}}(G) = \Gamma[u_1] \otimes_\vartheta \bar{\mathrm{M}}_G$, where, with an abuse of the notation ϑ, the composite of ϑ with the obvious surjection

$$\alpha\colon \mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG} (\mathrm{M}_G) \to \mathrm{end}_{\bar{\mathrm{M}}(N)} (\bar{\mathrm{M}}_G)$$

is denoted by $\vartheta\colon \Gamma[u_1] \to \mathrm{end}_{\bar{\mathrm{M}}(N)} (\bar{\mathrm{M}}_G)$ also. This abuse of notation is justified by the observation that α induces the structure of a differential graded $\mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG} (\mathrm{M}_G)$-module on $\bar{\mathrm{M}}_G$. Now, in view of (8.3.1), the reduced operators $\vartheta(\gamma_i(u_1)) \in \mathrm{end}_{\bar{\mathrm{M}}(N)} (\bar{\mathrm{M}}_G)$ are given by

$$\vartheta(\gamma_1(u_1)) = k_1 \begin{bmatrix} 0 & 0 & -\gamma_1(u_y) & (\gamma_1(u_y))v_y \\ -l & 0 & \dfrac{l(l-1)}{2} v_y & \gamma_1(u_y) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -l & 0 \end{bmatrix}$$

and, for $i \geqslant 2$,

$$\vartheta(\gamma_i(u_1)) = k_1^i \begin{bmatrix} 0 & 0 & -\gamma_i(u_y) & (\gamma_i(u_y))v_y \\ 0 & 0 & \dfrac{l(l-1)}{2} (\gamma_{i-1}(u_y))v_y & \gamma_i(u_y) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Consequently, on the reduced object $\bar{\mathrm{M}}(G) = \Gamma[u_1] \otimes_\vartheta \bar{\mathrm{M}}_G$ the differential d is given by the formula

$$\begin{aligned} d((\gamma_i(u_1))a) &= (\gamma_i(u_1))\, da - (\vartheta \cap\ )(\gamma_i(u_1))a \\ &= (\gamma_i(u_1))\, da - \sum_{j=1}^{i} (\gamma_{i-j}(u_1))(\vartheta(\gamma_j(u_1))a), \end{aligned}$$

where $a \in \bar{\mathrm{M}}_G$, and where $da \in \bar{\mathrm{M}}_G$ refers to the differential in $\bar{\mathrm{M}}_G$. Explicitly, as an $\bar{\mathrm{M}}(N)$-module, $\bar{\mathrm{M}}_G$ is free with basis 1, v_1, v_2, $v_1 v_2$, and the only non-zero differential on the basis elements is given by the formula

$$d(v_1 v_2) = -v_y.$$

Hence, with reference to the differential graded $\bar{\mathrm{M}}(N)$-module structure on $\bar{\mathrm{M}}_G$, the differential d on $\bar{\mathrm{M}}(G)$ may explicitly be described by the formulas

$$\begin{aligned}
d(\gamma_i(u_1)) &= m_1((\gamma_{i-1}(u_1))v_1),\\
d((\gamma_i(u_1))v_1 &= 0,\\
d((\gamma_i(u_1))v_2) &= \sum_{j=1}^{i} \left(k_1^j(\gamma_{i-j}(u_1)) \left(\gamma_j(u_y) - \frac{l(l-1)}{2} v_1(\gamma_{j-1}(u_y))v_y \right)\right)\\
&\quad + m_1((\gamma_{i-1}(u_1))\, v_1 v_2) \qquad (8.4.1)\\
d((\gamma_i(u_1))\, v_1 v_2) &= -\left((\gamma_i(u_1))v_y + \sum_{j=1}^{i} k_1^j(\gamma_{i-j}(u_1))((\gamma_j(u_y))v_y + v_1(\gamma_j(u_y)))\right),
\end{aligned}$$

where $i \geqslant 1$ or $i \geqslant 0$ as appropriate.

Our next aim is to compute the integral cohomology of G from $\bar{\mathrm{M}}(G)$. To get our hands on it, we now give a description of the object $(\bar{\mathrm{M}}(G))^*$ dual to $\bar{\mathrm{M}}(G)$. It is of course straightforward to dualize the differential on $\bar{\mathrm{M}}(G)$ given in (8.4.1) above, but this is *not* illuminating and rather a mess. We therefore explain how the *bundle structure* on $(\bar{\mathrm{M}}(G))^*$ dual to the one on $\bar{\mathrm{M}}(G)$ may be used to *organise* the description of the differential on $(\bar{\mathrm{M}}(G))^*$. We remind the reader that under the present circumstances,

$$\bar{\mathrm{M}}(N) = \Gamma[u_y] \otimes_{\tau_N} \Lambda[v_y], \qquad \text{where} \quad \tau_N(u_y) = l v_y.$$

By construction, the differential on $\bar{\mathrm{M}}(G) = \Gamma[u_1] \otimes_\vartheta \bar{\mathrm{M}}_G$ is compatible with the induced left and right $(\Gamma[u_1])$-comodule and $(\bar{\mathrm{M}}(N))$-module structures, respectively. Since as an induced $(\bar{\mathrm{M}}(N))$-module, $\bar{\mathrm{M}}_G = \Lambda[v_1, v_2] \otimes \bar{\mathrm{M}}(N)$, the appropriate adjoint of

$$\vartheta\colon \Gamma[u_1] \to \mathrm{end}_{\bar{\mathrm{M}}(N)}(\bar{\mathrm{M}}_G) \subseteq \mathrm{Hom}_{\bar{\mathrm{M}}(N)}(\Lambda[v_1, v_2] \otimes \bar{\mathrm{M}}(N), \bar{\mathrm{M}}_G)$$

way be written

$$\Phi_\vartheta\colon \Gamma[u_1] \otimes \Lambda[v_1, v_2] \to \bar{\mathrm{M}}_G.$$

Explicitly, it is given by the formulas

$$\Phi_9(\gamma_i(u_1)) = \begin{cases} m_1 v_1, & \text{if } i = 1, \\ 0, & \text{if } i \geqslant 2, \end{cases}$$

$$\Phi_9((\gamma_i(u_1))v_1) = 0, \qquad \text{if } i \geqslant 0, \tag{8.4.2}$$

$$\Phi_9((\gamma_i(u_1))v_2) = \begin{cases} k_1\left(\gamma_1(u_y) - \dfrac{l(l-1)}{2} v_1 v_y\right) + m_1 v_1 v_2, & \text{if } i = 1, \\ k_1^i\left(\gamma_i(u_y) - \dfrac{l(l-1)}{2} v_1(\gamma_{i-1}(u_y))v_y\right), & \text{if } i \geqslant 2, \end{cases}$$

$$\Phi_9((\gamma_i(u_1))\, v_1 v_2) = k_1^i((\gamma_i(u_y))v_y + v_1(\gamma_i(u_y)), \qquad i \geqslant 0,$$

where as before the tensor product symbol has been discarded. Notice that Φ_9 may as well be characterized as the composite

$$\Gamma[u_1] \otimes \Lambda[v_1, v_2] \longrightarrow \bar{\mathrm{M}}(G) \xrightarrow{d} \bar{\mathrm{M}}(G) \longrightarrow \bar{\mathrm{M}}_G,$$

where the unlabelled arrows are the obvious injection and projection maps. Consequently, with reference to the duals c_1, ω_1, ω_2, c_y, ω_y of respectively u_1, v_1, v_2, u_y, v_y in the bases of monomials, the dual object $(\bar{\mathrm{M}}(G))^*$ looks like

$$\begin{aligned}(\bar{\mathrm{M}}(G))^* &= \mathrm{P}[c_1] \otimes_{9^*} (\bar{\mathrm{M}}_G)^* \\ &= \mathrm{P}[c_1] \otimes_{9^*} (\Lambda[\omega_1, \omega_2] \otimes_{\xi^*} (\mathrm{P}[c_y] \otimes_{\tau_N^*} \Lambda[\omega_y])),\end{aligned}$$

and its differential d is compatible with the induced left and right $(\mathrm{P}[c_1])$-module and $((\bar{\mathrm{M}}(N))^*)$-comodule structures on $(\bar{\mathrm{M}}(G))^*$, whence it is completely determined by the morphism Φ_9^* dual to Φ_9 and the differential (say) d_{G_F} on

$$(\bar{\mathrm{M}}_G)^* = (\bar{\mathrm{M}}(G_F))^* = \Lambda[\omega_1, \omega_2] \otimes_{\xi^*} (\mathrm{P}[c_y] \otimes_{\tau_N^*} \Lambda[\omega_y]);$$

the latter has been given in (7.3). Notice that the morphism Φ_9^* may be characterised as the composite

$$\Phi_9^*\colon (\bar{\mathrm{M}}_G)^* \longrightarrow (\bar{\mathrm{M}}(G))^* \xrightarrow{d} (\bar{\mathrm{M}}(G))^* \longrightarrow \mathrm{P}[c_1] \otimes (\Lambda[\omega_1, \omega_2]),$$

where again the unlabelled arrows are the obvious injection and projection maps. Now (8.4.2) implies at once that Φ_9^* is given by the formulas

$$\Phi_9^*(\omega_1) = m_1 c_1,$$

$$\Phi_9^*(\omega_1 c_y^i) = k_1^i c_1^i \omega_1 \omega_2, \qquad i \geqslant 1,$$

$$\Phi_9^*(\omega_1 c_y^i \omega_y) = -k_1^{i+1} \frac{l(l-1)}{2} c_1^{i+1} \omega_2, \qquad i \geqslant 0,$$

$$
\begin{aligned}
\Phi_{\vartheta}^{*}(c_y^i) &= k_1^i c_1^i \omega_2, && i \geqslant 1,\\
\Phi_{\vartheta}^{*}(c_y^i \omega_y) &= k_1^i c_1^i \omega_1 \omega_2, && i \geqslant 1,\\
\Phi_{\vartheta}^{*}(\omega_2 c_y^i) &= 0, && i \geqslant 0,\\
\Phi_{\vartheta}^{*}(\omega_2 c_y^i \omega_y) &= 0, && i \geqslant 0,\\
\Phi_{\vartheta}^{*}(\omega_1 \omega_2) &= m_1 c_1 \omega_2, && \\
\Phi_{\vartheta}^{*}(\omega_1 \omega_2 c_y^i) &= 0, && i \geqslant 1,\\
\Phi_{\vartheta}^{*}(\omega_1 \omega_2 c_y^i \omega_y) &= 0, && i \geqslant 0. \qquad (8.4.3)
\end{aligned}
$$

Inspection shows that Φ_{ϑ}^{*} is compatible with the obvious $(\Lambda[\omega_2])$-module structure, but it is in general *not* compatible with the obvious $(\Lambda[\omega_1, \omega_2])$-module structure; see (8.8) below. We now write α_{ϑ} for the composite

$$
\begin{aligned}
\alpha_{\vartheta} \colon (\bar{\mathrm{M}}_G)^* &\xrightarrow{\Delta} (\bar{\mathrm{M}}_G)^* \otimes (\bar{\mathrm{M}}(N))^*\\
&\xrightarrow{\Phi_{\vartheta}^{*} \otimes (\bar{\mathrm{M}}(N))^*} \mathrm{P}[c_1] \otimes (\Lambda[\omega_1, \omega_2]) \otimes (\bar{\mathrm{M}}(N))^*,
\end{aligned}
$$

where Δ is the $(\bar{\mathrm{M}}(N))^*$-comodule structure which is dual to the induced $(\bar{\mathrm{M}}(N))$-module structure on $(\bar{\mathrm{M}}_G)$. Then the effect of the differential d on the basis elements b of $(\bar{\mathrm{M}}(G))^*$ as a $(\mathrm{P}[c_1] \otimes \Lambda[\omega_2])$-module is given by the formula

$$d(b) = d_{G_F}(b) + \alpha_{\vartheta}(b).$$

To compute it explicitly, we systematically use the Eilenberg–Koszul convention already mentioned in (7.3), and we obtain the formulas

$$d(c_y^i) = \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_2 c_y^{i-j}, \qquad i \geqslant 1,$$

$$d(c_y^i \omega_y) = (\omega_1 \omega_2 c_y^i - l c_y^{i+1}) + \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j (\omega_1 \omega_2 + \omega_2 \omega_y) c_y^{i-j}, \qquad i \geqslant 0,$$

$$d(\omega_1) = m_1 c_1,$$

$$d(\omega_1 c_y^i) = m_1 c_1 c_y^i - \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_1 \omega_2 c_y^{i-j}, \qquad i \geqslant 1,$$

$$d(\omega_1 c_y^i \omega_y) = \begin{cases} l \omega_1 c_y^{i+1} + k_1 \dfrac{l(l-1)}{2} c_1 \omega_2 c_y^i + m_1 c_1 c_y^i \omega_y \\ + \displaystyle\sum_{j=1}^{i} \binom{i}{j} k_1^j \left(k_1 \frac{l(l-1)}{2} c_1^{j+1} \omega_2 - c_1^j \omega_1 \omega_2 \omega_y \right) c_y^{i-j}, \end{cases} \qquad i \geqslant 0.$$

We now take the ground ring R to be the integers. For convenience, we shall write

$$l\{i\} = \left(l, i, \binom{i}{2} k_1, \binom{i}{3} k_1^2, \ldots, \binom{i}{j} k_1^{j-1}, \ldots, k_1^{i-1}\right), \qquad i \geq 1, \quad (8.4.5)$$

where as usual $(\ldots)$ denotes the greatest common divisor. Inspection of (8.4.4) shows at once that the following cochains are cocycles over the integers:

$$\begin{aligned} \zeta_{2i+1} &= \omega_2 c_y^i, && i \geq 0, \\ \zeta_{2i} &= \frac{l}{l\{i\}} c_y^i - \sum_{j=1}^{i} \frac{\binom{i}{j} k_1^{j-1}}{l\{i\}} c_1^{j-1} \omega_1 \omega_2 c_y^{i-j}, && i \geq 1, \qquad (8.4.6) \\ \omega_3 &= k_1 c_1 \omega_y + \omega_1 c_y. \end{aligned}$$

Abusing notation, we shall denote the classes of these cocycles by the same symbols. Under appropriate circumstances more classes are provided by the following.

LEMMA 8.4.7. *Let p be a prime which divides l. Then*

$$\sigma_{2p} = \frac{l}{p} c_y^p - \sum_{j=1}^{p-1} \frac{\binom{p}{j}}{p} k_1^{j-1} c_1^{j-1} \omega_1 \omega_2 c_y^{p-j} - \frac{l}{p} k_1^{p-1} c_1^{p-1} c_y$$

is a cocycle. Moreover, if p does not divide k_1, $p\sigma_{2p} = \zeta_{2p} \in \mathrm{H}^*(G, \mathbf{Z})$, while if p divides k_1, $\sigma_{2p} = \zeta_{2p} \in \mathrm{H}^*(G, \mathbf{Z})$.

Proof. A calculation shows that σ_{2p} is a cocycle, and that, on the cochain level,

$$p\sigma_{2p} = \zeta_{2p} + k_1^{p-1} c_1^{p-1} (\omega_1 \omega_2 - l c_y) = \zeta_{2p} + d(k_1^{p-1} c_1^{p-1} \omega_y)$$

or

$$\sigma_{2p} = \zeta_{2p} + k_1^{p-1} c_1^{p-1} (\omega_1 \omega_2 - l c_y) = \zeta_{2p} + d(k_1^{p-1} c_1^{p-1} \omega_y)$$

as appropriate. This proves the lemma. ∎

THEOREM 8.4.8. *Suppose that $l = p_1 p_2 \cdots p_v$, where $p_1, p_2, \ldots, p_v$ are pairwise distinct prime numbers, and that k_1 and l are relatively prime. Then, as a graded commutative algebra, the integral cohomology algebra of G is generated by*

$$c_1, \xi_1, \xi_3, \ldots, \sigma_{2p}, \omega_3,$$

where $p \in P = \{p_1, \ldots, p_v\}$, subject to the relations

$$\sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \xi_{2(i-j)+1} = 0, \qquad i \geqslant 1, \tag{8.4.9}$$

$$m_1 c_1 = 0, \tag{8.4.10}$$

$$p\sigma_{2p} = \sum_{j=1}^{p-1} \binom{p-1}{j} k_1^{j-1} c_1^{j-1} \xi_{2(p-1-j)+1} \omega_3, \tag{8.4.11}$$

$$\xi_{2i+1} \xi_{2j+1} = 0, \qquad i, j \geqslant 0, \tag{8.4.12}$$

$$\xi_{2j+1} \sigma_{2p} = \frac{l}{p} (\xi_{2(j+p)+1} - k_1^{p-1} c_1^{p-1} \xi_{2(j+1)+1}), \qquad j \geqslant 0, \tag{8.4.13}$$

$$\sigma_{2p} \sigma_{2p'} = \frac{l}{pp'} \sum_{j=1}^{p+p'-1} \binom{p+p'-1}{j} k^{j-1} c_1^{j-1} \xi_{2(p+p'-1-j)+1} \omega_3, \tag{8.4.14}$$

where $p \neq p'$,

$$\omega_3^2 = \begin{cases} 0, & \textit{if } l \textit{ is odd}, \\ k_1 c_1 \sigma_4 + A c_1 \xi_1 \omega_3 + B \xi_3 \omega_3, & \textit{if } l \textit{ is even}, \end{cases} \tag{8.4.15}$$

$$l\xi_{2i+1} = 0, \qquad i \geqslant 1, \tag{8.4.16}$$

$$l\omega_3 = 0, \tag{8.4.17}$$

where $p, p' \in P$, and where A and B are appropriate (possibly trivial) integers.

The proof will be given in (8.5)–(8.7) below. The reader will note that that under the circumstances of (8.4.8) the integral cohomology algebra of G is finitely generated and admits a finite set of defining relations. In fact, a standard argument involving the Chinese reminder theorem shows that $\mathrm{H}^*(G, \mathbf{Z})$ is generated by

$$c_1, \xi_1, \xi_3, \ldots, \xi_{2l-1}, \sigma_{2p}, \omega_3,$$

where $p \in P$. Furthermore, it is straightforward, albeit a bit messy, to write out a finite set of relations, and we spare the reader and ourselves these added troubles. However, it seems worthwhile mentioning the following special case:

THEOREM 8.4.8.p. *Suppose l is a prime p which does not divide k_1. Then, as a graded commutative algebra, the integral cohomology algebra of G is generated by*

$$c_1, \xi_1, \xi_3, \ldots, \xi_{2p-1}, \sigma_{2p}, \omega_3,$$

subject to the relations

$$\sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \zeta_{2(i-j)+1} = 0, \qquad 1 \leqslant i \leqslant p, \tag{8.4.9.p}$$

$$m_1 c_1 = 0, \tag{8.4.10.p}$$

$$p\sigma_{2p} = \sum_{j=1}^{p-1} \binom{p-1}{j} k_1^{j-1} c_1^{j-1} \xi_{2(p-1-j)+1} \omega_3, \tag{8.4.11.p}$$

$$\omega_3^2 = \begin{cases} 0, & \text{if } l \text{ is odd}, \\ k_1 c_1 \sigma_4 + A c_1 \zeta_1 \omega_3 + B \xi_3 \omega_3, & \text{if } l = 2, \end{cases} \tag{8.4.15.p}$$

$$l\zeta_{2i+1} = 0, \qquad 1 \leqslant i \leqslant p, \tag{8.4.16.p}$$

$$l\omega_3 = 0, \tag{8.4.17.p}$$

where A and B are appropriate (possibly trivial) integers.

The cases where higher p'th powers occur in l or where l and k_1 are *not* relatively prime, or both, lead to subtle problems in invariant theory. We shall explain this in (8.7) below.

We mention that the numbering of the relations in (8.4.8) above has been chosen in a way which is consistent with (8.6.8) below. We also mention that in (8.4.8) the classes ζ_{2i}, cf. (8.4.6), play no role, but their significance will emerge in (8.7) below.

The computation of the mod p cohomology of G for any prime p is easy; we leave the details to the reader.

8.5. *The Proof of* (8.4.8) *except Relations* (8.4.14)

For the moment we do *not* assume the hypothesis of (8.4.8). We begin with the observation that the subgroup $(\mathbf{Z}/l) \times (\mathbf{Z}/m_1)$ of G generated by y and x_1 is manifestly normal in G, whence G fits also into an extension

$$1 \to (\mathbf{Z}/l) \times (\mathbf{Z}/m_1) \to G \to \mathbf{Z} \to 1, \tag{8.5.1}$$

where the quotient group $\mathbf{Z}$ is free cyclic generated by x_2. The cohomology spectral sequence of the latter boils down to a short exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^1(\mathbf{Z}, \mathrm{H}^{*-1}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1))) \\ &\to \mathrm{H}^*(G) \to \mathrm{H}^0(\mathbf{Z}, \mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1))) \to 0 \end{aligned} \tag{8.5.2}$$

of rings; here we have written $\mathrm{H}^*(-) = \mathrm{H}^*(-, \mathbf{Z})$ for short, and cohomology must be computed with respect to the induced action of $\mathbf{Z}$ coming from the extension (8.5.1). We mention that this action is actually hidden in the formula (8.4.4) for the differential on $(\bar{\mathrm{M}}(G))^*$. We spell out this action in (8.5.22) below. We mention in passing that the long exact sequence of (8.5.1) may also be obtained as the Wang sequence of a fibre bundle over S^1 modelling the extension (8.5.1). It is perhaps worthwhile emphasizing that while the computations of the left- and right-hand sides of (8.5.2) do not require any new machinery, the exact sequence (8.5.2) comes with an additive and multiplicative extension problem. The above relation (8.4.11) shows that even the additive extension problem is in general non-trivial. As far as the present example is concerned, it is precisely at this stage that our approach pays off, since it enables us to solve the extension problem.

We now rewrite $(\bar{\mathrm{M}}(G))^*$ in a way which reflects the extension (8.5.1): Let

$$(\bar{\mathrm{M}}((\mathbf{Z}/l)\times(\mathbf{Z}/m_1)))^* = (\mathrm{P}[c_1]\otimes\Lambda[\omega_1]\otimes\mathrm{P}[c_y]\otimes\Lambda[\omega_y], d) \tag{8.5.3}$$

be the dual of the reduced object $\bar{\mathrm{M}}((\mathbf{Z}/l)\times(\mathbf{Z}/m_1))$ for $(\mathbf{Z}/l)\times(\mathbf{Z}/m_1)$ introduced in [49, (3.2.8)]; we recall from [49, (3.0)] that it is actually a differential graded algebra generated by the four indicated generators, that c_1 and c_y are cycles, and that the differential d is determined by the formulas

$$d(\omega_1) = m_1 c_1, \qquad d(\omega_y) = l c_y. \tag{8.5.4}$$

For the sake of clarity, we spell out the following.

PROPOSITION 8.5.5. *As a graded commutative algebra, the integral cohomology algebra of* $(\mathbf{Z}/l)\times(\mathbf{Z}/m_1)$ *is generated by* c_1, ω_3, *and* c_y, *viewed as classes in* $(\bar{\mathrm{M}}((\mathbf{Z}/l)\times(\mathbf{Z}/m_1)))^*$, *subject to the relations*

$$m_1 c_1 = 0, \tag{8.5.6}$$

$$l c_y = 0, \tag{8.5.7}$$

$$l\omega_3 = 0, \tag{8.5.8}$$

$$\omega_3^2 = k_1^2 \frac{l(l-1)}{2} c_1^2 c_y + \frac{m_1(m_1-1)}{2} c_1 c_y^2. \tag{8.5.9}$$

It is clear that the obvious surjection

$$(\bar{\mathrm{M}}(G))^* \to (\bar{\mathrm{M}}((\mathbf{Z}/l)\times(\mathbf{Z}/m_1)))^* \tag{8.5.10}$$

which identifies generators having the same names is a comparison map for the restriction map

$$\mathrm{res}\colon \mathrm{H}^*(G, \mathbf{Z}) \to \mathrm{H}^*((\mathbf{Z}/l)\times(\mathbf{Z}/m_1), \mathbf{Z}). \tag{8.5.11}$$

Hence

$$\operatorname{res}(\xi_{2i+1}) = 0, \qquad i \geqslant 0, \tag{8.5.12}$$

$$\operatorname{res}(\zeta_{2i}) = \frac{l}{l\{i\}} c_y^2, \qquad i \geqslant 2, \tag{8.5.13}$$

$$\operatorname{res}(\sigma_{2p}) = \frac{l}{p} c_y^p - \frac{l}{p} k_1^{p-1} c_1^{p-1} c_y, \tag{8.5.14}$$

$$\operatorname{res}(c_1) = c_1, \tag{8.5.15}$$

$$\operatorname{res}(\omega_3) = \omega_3, \tag{8.5.16}$$

where as before we do not distinguish in notation between the classes $c_1 \in \mathrm{H}^2(G, \mathbf{Z})$ and $\omega_3 \in \mathrm{H}^3(G, \mathbf{Z})$ and their images in $\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})$.

Inspection shows that $(\bar{\mathrm{M}}(G))^*$ may be written

$$(\bar{\mathrm{M}}(G))^* = (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \,\omega_2 \oplus_{d_1} (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^*, \tag{8.5.17}$$

where d_1 is an operator

$$d_1 : (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \to (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \,\omega_2$$

which as a morphism of graded $\mathrm{P}[c_1]$-modules is given by those terms in (8.4.4) which raise ω_2-degree. In particular, as a differential graded $\mathrm{P}[c_1]$-module, the object $(\bar{\mathrm{M}}(G))^*$ is (decreasingly) filtered with

$$F^0((\bar{\mathrm{M}}(G))^*) = (\bar{\mathrm{M}}(G))^*, \qquad F^1((\bar{\mathrm{M}}(G))^*) = (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \,\omega_2,$$

the operator d_1 raises filtration by one, and there results a short exact sequence

$$0 \to (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \,\omega_2 \to (\bar{\mathrm{M}}(G))^* \to (\bar{\mathrm{M}}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1)))^* \to 0 \tag{8.5.18}$$

of chain complexes over $\mathrm{P}[c_1]$. Abusing notation, we shall denote the morphism

$$d_1 : \mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}) \to (\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}))\,\omega_2$$

induced by the operator d_1 by the same symbol as indicated. Inspection shows that as a morphism of graded $\mathrm{H}^*(\mathbf{Z}/m_1, \mathbf{Z}) = (\mathrm{P}[c_1]/m_1 c_1)$-modules,

$$\begin{aligned} d_1(c_y^i) &= \left(\sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j c_y^{i-j} \right) \omega_2, && i \geqslant 1, \\ d_1(\omega_3) &= 0, && \qquad (8.5.19) \\ d_1(\omega_3 c_y^i) &= \omega_3 \left(\sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j c_y^{i-j} \right) \omega_2, && i \geqslant 1. \end{aligned}$$

Now the long exact cohomology sequence of (8.5.18) boils down to a short exact sequence

$$0 \to \operatorname{coker}(d_1) \to \mathrm{H}^*(G, \mathbf{Z}) \to \ker(d_1) \to 0, \tag{8.5.20}$$

and this is just (8.5.2) above. In fact, by construction the quotient group $\mathbf{Z} = \langle x_2 \rangle$ of (8.5.1) acts on $(\mathbf{Z}/l) \times (\mathbf{Z}/m_1)$ by means of the rules

$$x_2 \cdot y = y, \qquad x_2 \cdot x_1 = x_1 y. \tag{8.5.21}$$

Furthermore, it acts on

$$\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}) = (\mathrm{P}[c_y, c_1] \otimes \Lambda[\omega_3])/(m_1 c_1, l c_y, l\omega_3)$$

by means of

$$x_2 \cdot c_1 = c_1, \qquad x_2 \cdot c_y = c_y + k_1 c_1, \qquad x_2 \cdot \omega_3 = \omega_3. \tag{8.5.22}$$

While it is not hard to prove this directly, we mention that a little thought reveals that (8.5.22) follows easily from the formulas (8.4.4) for the differential on $(\bar{\mathrm{M}}(G))^*$. With this action, the operator

$$d_1 \colon \mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}) \to \mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})\, \omega_2$$

may obviously be rewritten

$$d_1(c_y^i) = (x_2 \cdot c_y^i - c_y^i)\, \omega_2. \tag{8.5.23}$$

Hence $\operatorname{coker}(d_1) = \mathrm{H}^1(\mathbf{Z}, \mathrm{H}^{*-1}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}))$ and $\ker(d_1) = \mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})^{\mathbf{Z}}$ as asserted.

We now begin with the proof of (8.4.8); for clarity we mention that at present there is still no need to assume the hypothesis of (8.4.8): It is easy to see that (8.4.12) and (8.4.13) are true: In fact, with respect to the filtration on $\mathrm{H}^*(G, \mathbf{Z})$ induced by the one introduced above these relations hold on the associated graded object; however, since the elements ξ_{2i+1} have filtration 1, and since there is nothing left in higher filtration, it follows at once that there is no multiplicative extension problem for products of the kind $\xi_{2i+1}\xi_{2j+1}$ and $\xi_{2j+1}\sigma_{2p}$. Furthermore, the relation (8.4.15) does not present a problem either. In fact, if l is odd, ω_3^2 is zero for formal reasons. If l is even, we invoke (8.5.9), (8.5.13), and (8.5.14), and obtain

$$\begin{aligned}\omega_3^2 &= k_1^2 \frac{l(l-1)}{2} c_1^2 c_y + \frac{m_1(m_1-1)}{2} c_1 c_y^2 \in \mathrm{H}^6((\mathbf{Z}/m_1) \times (\mathbf{Z}/l), \mathbf{Z}) \\ &= k_1 c_1 \frac{l}{2} (k_1 c_1 c_y + c_y^2) = \begin{cases} \operatorname{res}(k_1 c_1 \zeta_4), & \text{if } k_1 \text{ is even}, \\ \operatorname{res}(k_1 c_1 \sigma_4), & \text{if } k_1 \text{ is odd}, \end{cases}\end{aligned}$$

whence the relation (8.4.15) holds.

Next we observe that in view of (8.5.6) and (8.5.19) the object

$$\operatorname{coker}(d_1) = \mathrm{H}^1(\mathbf{Z}, \mathrm{H}^{*-1}((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}))$$

is the ideal in $\mathrm{H}^*(G, \mathbf{Z})$ generated by the elements $\xi_{2i+1} = c_y^i \omega_2 \in \mathrm{H}^*(G, \mathbf{Z})$, and it is easy to see that its structure is entirely taken care of by the relations given in the theorem.

It is clear that $m_1 c_1 = 0$, i.e., that the relation (8.4.10) holds. We now prove that the relations (8.4.11) and (8.4.17) hold:

LEMMA 8.5.24. *In the general case, i.e., without the hypothesis in* (8.4.8), *for* $i \geqslant 1$,

$$l\{i+1\}\,\zeta_{2i+2} = \sum_{j=1}^{i} \binom{i}{j} k_1^{j-1} c_1^{j-1} \xi_{2(i-j)+1}\,\omega_3.$$

Proof. Recall that the generators ζ_{2i} were introduced in (8.4.6). From (8.4.4) we deduce that, for $i \geqslant 1$,

$$\begin{aligned}
d(c_y^i \omega_y) &= (\omega_1 \omega_2 c_y^i - l c_y^{i+1}) + \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j (\omega_1 \omega_2 + \omega_2 \omega_y)\, c_y^{i-j} \\
&= \omega_1 \omega_2 c_y^i + \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_1 \omega_2 c_y^{i-j} \\
&\quad + \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_2 \omega_y c_y^{i-j} - l c_y^{i+1} \\
&= -\Bigg(l c_y^{i-1} - \omega_1 \omega_2 c_y^i - \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_1 \omega_2 c_y^{i-j} \\
&\quad - \sum_{j=1}^{i} \binom{i}{j} k_1^{j-1} c_1^{j-1} \omega_1 \omega_2 c_y^{i-j+1} \Bigg) \\
&\quad - (k_1 c_1 \omega_y + \omega_1 c_y) \sum_{j=1}^{i} \binom{i}{j} k_1^{j-1} c_1^{j-1} \omega_2 c_y^{i-j} \\
&= -l\{i+1\}\,\zeta_{2i+2} - \omega_3 \sum_{j=1}^{i} \binom{i}{j} k_1^{j-1} c_1^{j-1} \xi_{2(i-j)+1}.
\end{aligned}$$

We mention, for clarity, that the above products should be understood in the naive sense, i.e., they arise by juxtaposition of monomials as appropriate. ▮

Under the circumstances of (8.4.8) we have $l\{i+1\} = 1$. Now, taking

$i=p-1$, in view of (8.4.7) we conclude from the lemma at once that the relations (8.4.11) hold. Next, again from (8.4.4) we deduce

$$d(\omega_1\omega_y)=l\omega_1 c_y+m_1 c_1\omega_y=l(\omega_1 c_y+k_1 c_1\omega_y)=l\omega_3,$$

whence (8.4.17) holds as well.

Next we observe that the relations (8.4.14) manifestly hold modulo elements of filtration 1, as inspection of the effect of the restriction map (8.5.13) on the products on the left-hand side of these relations shows. We now consider the graded module which underlies the graded algebra generated abstractly by the indicated generators, listed in (8.4.8), subject to the relations (8.4.9)–(8.4.17). In view of what has already been proved, the obvious map of this module into $\mathrm{H}^*(G, \mathbf{Z})$ identifies the ideal generated by the ξ_{2i+1} with the term $\mathrm{coker}(d_1)$ in (8.5.20). However, modulo this ideal, the resulting quotient algebra is the graded algebra generated abstractly by c_1, σ_{2p}, for $p\in P$, ω_3, subject to the relations

$$\begin{aligned} m_1c_1&=0, && \\ p\sigma_{2p}&=0, && p\in P,\\ \sigma_{2p}\sigma_{2p'}&=0, && p\neq p',\\ \omega_3^2&=\begin{cases}0, & \text{if } l \text{ is odd},\\ k_1c_1\sigma_4, & \text{if } l \text{ is even},\end{cases} \end{aligned}\tag{8.5.25}$$

and a little thought reveals that this is actually just the subalgebra of the invariants $(\mathrm{H}^*((\mathbf{Z}/l)\times(\mathbf{Z}/m_1), \mathbf{Z}))^{\mathbf{Z}}$ generated by the classes c_1, $c_y^p-k_1^{p-1}c_1^{p-1}c_y$, for $p\in P$, ω_3. To complete the proof, we reproduce some standard invariant theory, which is classical (Dickson [44], Feldstein [45], Turner [61]). More recent references are Lewis [52, Lemma 6.28], Mui [54], and Steinberg [59].

LEMMA 8.5.26. *Let p be a prime, let $\mathbf{F}_p$ be the field with p elements, and let the generator of the group $\mathbf{Z}/p$ act on the polynomial algebra $\mathbf{F}_p[z_1, z_2]$ by means of the rule*

$$z_1\mapsto z_1, \qquad z_2\mapsto z_2+z_1.$$

Then the algebra $(\mathbf{F}_p[z_1, z_2])^{\mathbf{Z}/p}$ of invariants is the polynomial algebra generated by z_1 and $z_2^p-z_1^{p-1}z_2$.

Proof. Notice first that $z_2^p-z_1^{p-1}z_2$ is indeed invariant. Further, as a $(\mathbf{F}_p[z_1, z_2^p-z_1^{p-1}z_2])$-module, the algebra $\mathbf{F}_p[z_1, z_2]$ is manifestly free with basis $1, z_2, z_2^2, \dots, z_2^{p-1}$, and from this description one readily verifies that indeed

$$(\mathbf{F}_p[z_1, z_2])^{\mathbf{Z}/p} \subseteq (\mathbf{F}_p[z_1, z_2^p - z_1^{p-1} z_2])$$

as asserted. ▌

The proof of (8.4.8) except relations (8.4.14) is now completed by the following.

PROPOSITION 8.5.27. *Under the circumstances of* (8.4.8) *the subalgebra of invariants is as an algebra generated by the classes* $c_1, c_y^p - k_1^{p-1} c_1^{p-1} c_y, \omega_3$, *where* $p \in P = \{p_1, \dots, p_v\}$.

Proof. In fact, let p be a prime in P, and localize at p. Then, with $z_1 = c_1$ and $z_2 = k_1 c_y$, the cohomology ring of $(\mathbf{Z}/l) \times (\mathbf{Z}/m_1)$ may be written

$$\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}_{(p)}) = \mathbf{F}_p[z_1, z_2] \oplus (\mathbf{F}_p[z_1, z_2])\, \omega_3,$$

and the action of $\mathbf{Z}$ is compatible with the indicated direct sum decomposition and factors through an action of $\mathbf{Z}/p$. In view of (8.5.26), we conclude that

$$(\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}_{(p)}))^{\mathbf{Z}}$$

is as an algebra generated by $c_1, c_y^p - k_1^{p-1} c_1^{p-1} c_y, \omega_3$.

Likewise, if we localize away from $l = p_1 \cdots p_v$,

$$\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}[1/l]) = \mathrm{H}^*(\mathbf{Z}/m_1, \mathbf{Z}[1/l]),$$

and the entire cohomology ring is invariant. In view of the Chinese reminder theorem, the algebra of invariants has the indicated generators over the integers. ▌

Hence $\mathrm{H}^*(G, \mathbf{Z})$ has the asserted structure, except that we still have to verify (8.4.14). It seems worthwhile mentioning that, since by construction $\bar{\mathrm{M}}_G = \bar{\mathrm{M}}(G_F)$,

$$\bar{\mathrm{M}}(G) = \Gamma[u_1] \otimes_{\mathfrak{g}} \bar{\mathrm{M}}_G = \Gamma[u_1] \otimes_{\mathfrak{g}} \bar{\mathrm{M}}(G_F),$$

the obvious injection $\bar{\mathrm{M}}(G_F) \to \bar{\mathrm{M}}(G)$ is a comparison map for the projection map $G_F \to G$, and hence it is straightforward to spell out the inflation map $\mathrm{H}^*(G, \mathbf{Z}) \to \mathrm{H}^*(G_F, \mathbf{Z})$. We leave the details to the reader.

8.6. *A Digression and the Relations* (8.4.14)

Our present aim is to verify the relations (8.4.14). We note that these relations are the solution of a multiplicative extension problem. At first we make a digression which we believe to be interesting in its own right. We return to the situation where the numbers l and k_1 are arbitrary, i.e., where no hypothesis of the kind in (8.4.8) is required:

Inspection shows that, if we write $\tilde{\partial}$ for the restriction of the differential, the object

$$\tilde{\mathrm{M}}(G) = (\mathrm{P}[c_1] \otimes \Lambda[\omega_1, \omega_2] \otimes \mathrm{P}[c_y], \tilde{\partial}) \tag{8.6.1}$$

is a differential graded $(\mathrm{P}[c_1] \otimes \Lambda[\omega_2])$-submodule of $(\bar{\mathrm{M}}(G))^*$, and the latter fits into the short exact sequence

$$0 \to \tilde{\mathrm{M}}(G) \to (\bar{\mathrm{M}}(G))^* \to (\tilde{\mathrm{M}}(G))\, \omega_y \to 0 \tag{8.6.2}$$

of differential graded $(\mathrm{P}[c_1] \otimes \Lambda[\omega_2])$-modules; still by inspection we see that on the quotient $(\tilde{\mathrm{M}}(G))\, \omega_y = (\tilde{\mathrm{M}}(G)) \otimes \mathbf{Z}\omega_y$ the inherited differential boils down to $d_{\tilde{\mathrm{M}}(G)} \otimes (\mathbf{Z}\omega_y)$.

Before we proceed further we pause and explain briefly what is behind the short exact sequence (8.6.2): Embed $N = \mathbf{Z}/l$ into the circle group S^1 in the standard way, i.e., by means of the rule $y \mapsto \mathrm{e}^{2\pi i/l}$. This obviously determines a non-connected Lie group $\tilde{G}$ which fits into a central extension

$$\tilde{\mathrm{e}} : 0 \to S^1 \to \tilde{G} \to Q \to 1 \tag{8.6.3}$$

of S^1 by Q, so that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & S^1 & \longrightarrow & \tilde{G} & \longrightarrow & Q & \longrightarrow & 1. \end{array} \tag{8.6.4}$$

Moreover, a classifying space BG may be written as the total space of an S^1-bundle

$$S^1 \to BG \to B\tilde{G}. \tag{8.6.5}$$

It can be shown that $\tilde{\mathrm{M}}(G)$ is a model for the cochains of $B\tilde{G}$, and that the Gysin sequence of (8.6.5) is just the long exact cohomology sequence of (8.6.2). We refrain from giving the details, but it is conceptually useful to keep this fact in mind. Actually, we could concentrate at a single prime p at a time, i.e., we could assume $l = p^k$ for some k, and we could then work with the *Prüfer* group $\mathbf{Z}(p^\infty)$ instead of S^1; our methods in [49] cover this group and central extensions of it. We shall return to this elsewhere.

Let now $\Delta : \mathrm{M}(G) \to \mathrm{M}(G) \otimes \mathrm{M}(G)$ be the diagonal map for $\mathrm{M}(G)$ constructed in Subsection (4.5) of [49]; it is equivariant with respect to the induced $(\mathrm{M}(N) \otimes_{\mathbf{Z}N} \mathbf{Z}G)$-module structures, where $(\mathrm{M}(N) \otimes_{\mathbf{Z}N} \mathbf{Z}G)$ acts on $\mathrm{M}(G) \otimes \mathrm{M}(G)$ by means of its diagonal map. This implies that the dual morphism $(\bar{\mathrm{M}}(G))^* \otimes (\bar{\mathrm{M}}(G))^* \to (\bar{\mathrm{M}}(G))^*$ restricts to a morphism $\tilde{\mathrm{M}}(G) \otimes \tilde{\mathrm{M}}(G) \to \tilde{\mathrm{M}}(G)$ and hence induces a ring structure on $\mathrm{H}^*(\tilde{\mathrm{M}}(G))$, so that the induced morphism

$$\mathrm{H}^*(\tilde{\mathrm{M}}(G)) \to \mathrm{H}^*(G, \mathbf{Z}) \tag{8.6.6}$$

is a morphism of graded rings. The ring structure on $\mathrm{H}^*(\tilde{\mathrm{M}}(G))$ is of course just the ring structure of $\mathrm{H}^*(B\tilde{G}, \mathbf{Z})$; again we do not give the details.

We now compute the cohomology ring of the object $\tilde{\mathrm{M}}(G)$ under the circumstances of (8.4.8). We proceed in much the same way as we did in (8.5) above, and we shall indicate the similarity by using the same notation for cocycles, cohomology classes, etc., but we shall adorn them with a tilde whenever appropriate.

Viewed as cocycles, the elements ξ_{2i+1}, ζ_{2i}, and σ_{2p} are defined on $\tilde{\mathrm{M}}(G)$, viewed as a subcomplex of $(\bar{\mathrm{M}}(G))^*$. We denote the cohomology classes of $\tilde{\mathrm{M}}(G)$ represented by them respectively by $\tilde{\xi}_{2i+1}$, $\tilde{\zeta}_{2i}$, and $\tilde{\sigma}_{2p}$. Obviously the argument given for (8.4.7) also proves that, for any prime p that divides l but not k_1,

$$p\tilde{\sigma}_{2p} = \tilde{\zeta}_{2p} \in \mathrm{H}^*(\tilde{\mathrm{M}}(G)). \tag{8.6.7}$$

THEOREM 8.6.8. *Suppose that $l = p_1 p_2 \cdots p_v$, where $p_1, p_2, \ldots, p_v$ are pairwise distinct prime numbers, and that k_1 and l are relatively prime. Then, as a graded commutative algebra,* $\mathrm{H}^*(\tilde{\mathrm{M}}(G))$ *is generated by*

$$c_1, \tilde{\xi}_1, \tilde{\xi}_3, \ldots, \tilde{\zeta}_2, \tilde{\zeta}_4, \ldots, \tilde{\sigma}_{2p}, \qquad \text{where} \quad p \in P = \{p_1, \ldots, p_v\},$$

subject to the relations

$$\sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \tilde{\xi}_{2(i-j)+1} = 0, \qquad i \geqslant 1, \tag{8.6.9}$$

$$m_1 c_1 = 0, \tag{8.6.10}$$

$$p\tilde{\sigma}_{2p} = \tilde{\zeta}_{2p}, \tag{8.6.11}$$

$$\tilde{\xi}_{2i+1} \tilde{\xi}_{2j+1} = 0, \qquad i, j \geqslant 0, \tag{8.6.12}$$

$$\tilde{\xi}_{2j+1} \tilde{\sigma}_{2p} = \frac{l}{p} (\tilde{\xi}_{2(j+p)+1} - k_1^{p-1} c_1^{p-1} \tilde{\xi}_{2(j+1)+1}), \qquad j \geqslant 0, \tag{8.6.13}$$

$$\tilde{\sigma}_{2p} \tilde{\sigma}_{2p'} = \frac{l}{pp'} \tilde{\zeta}_{2(p+p')}, \qquad p \neq p', \tag{8.6.14}$$

$$\tilde{\zeta}_{2i}\tilde{\zeta}_{2j} = l\tilde{\zeta}_{2(i+j)}, \qquad i, j \geq 1, \tag{8.6.15}$$

$$\tilde{\xi}_{2j+1}\tilde{\zeta}_{2i} = l\tilde{\xi}_{2(j+i)+1}, \qquad i \geq 1, j \geq 0, \tag{8.6.16}$$

$$k_1 c_1 \tilde{\zeta}_{2i} = 0, \qquad i \geq 2, \tag{8.6.17}$$

where p, $p' \in P$.

In order to prove this, we rewrite the object $\tilde{\mathrm{M}}(G)$ in formally the same way as we rewrote $(\bar{\mathrm{M}}(G))^*$ in (8.5) above:

The subgroup $S^1 \times (\mathbf{Z}/m_1)$ of $\tilde{G}$ generated by S^1 and x_1 is manifestly normal in $\tilde{G}$, and hence the group $\tilde{G}$ fits also into an extension of the kind

$$1 \to S^1 \times (\mathbf{Z}/m_1) \to \tilde{G} \to \mathbf{Z} \to 1. \tag{8.6.18}$$

Let

$$\mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)) = (\mathrm{P}[c_1] \otimes \Lambda[\omega_1] \otimes \mathrm{P}[c_y], \tilde{d}), \tag{8.6.19}$$

where as a morphism of graded $(\mathrm{P}[c_1] \otimes \mathrm{P}[c_y])$-modules the differential $\tilde{d}$ is given by

$$\tilde{d}(\omega_1) = m_1 c_1.$$

Notice that $\mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1))$ computes the cohomology of $B(S^1 \times (\mathbf{Z}/m_1))$. Now

$$\tilde{\mathrm{M}}(G) = (\mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)))\,\omega_2 \oplus_{\tilde{d}_1} \mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)), \tag{8.6.20}$$

where $\tilde{d}_1$ is an operator

$$\tilde{d}_1 : \mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)) \to (\mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)))\,\omega_2$$

which is again induced by (8.4.4); hence, as a morphism of graded $(\mathrm{P}[c_1] \otimes \Lambda[\omega_1])$-modules, it is given by

$$\tilde{d}_1(c_y^i) = \sum_{j=1}^{i} \binom{i}{j} k_1^j c_1^j \omega_2 c_y^{i-j}, \qquad i \geq 1, \tag{8.6.21}$$

and there results a short exact sequence

$$0 \to (\mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)))\,\omega_2 \to \tilde{\mathrm{M}}(G) \to \mathrm{M}^*(S^1 \times (\mathbf{Z}/m_1)) \to 0 \tag{8.6.22}$$

of chain complexes. To write out its long exact cohomology sequence concisely, we observe first that, by construction, the quotient group $\mathbf{Z} = \langle x_2 \rangle$ of (8.6.18) acts on $S^1 \times (\mathbf{Z}/m_1)$ by means of the rules

$$x_2 \cdot y = y, \qquad x_2 \cdot x_1 = x_1 y, \qquad x_2 \cdot z = z, \qquad z \in S^1. \tag{8.6.23}$$

Furthermore, it acts on $H^*(B(S^1\times(\mathbf{Z}/m_1)),\mathbf{Z})=P[c_y,c_1]/m_1c_1$ by means of

$$x_2\cdot c_1=c_1,\qquad x_2\cdot c_y=c_y+k_1c_1,\qquad x_2\cdot\omega_3=\omega_3. \tag{8.6.24}$$

While it is not hard to prove this directly, we mention that again a little thought reveals that (8.6.24) follows easily from the formulas (8.4.4) for the differential on $(\tilde{\mathrm{M}}(G))^*$. With this action, the induced operator

$$\tilde{d}_1: H^*(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z})\to(H^*(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z}))\,\omega_2$$

may be written

$$\tilde{d}_1(c_y^i)=(x_2\cdot c_y^i-c_y^i)\,\omega_2. \tag{8.6.25}$$

Hence

$$\operatorname{coker}(\tilde{d}_1)=H^1(\mathbf{Z},H^{*-1}(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z})),$$
$$\ker(\tilde{d}_1)=H^*(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z})^{\mathbf{Z}},$$

and the long exact cohomology sequence of (8.6.22) boils down to

$$\begin{aligned}0&\to H^1(\mathbf{Z},H^{*-1}(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z}))\to H^*(\tilde{\mathrm{M}}(G))\\&\to H^*(B(S^1\times\mathbf{Z}/m_1),\mathbf{Z})^{\mathbf{Z}}\to 0.\end{aligned} \tag{8.6.26}$$

We mention that this is actually just the exact sequence

$$\begin{aligned}0&\to H^1(\mathbf{Z},H^{*-1}(B(S^1\times(\mathbf{Z}/m_1))))\to H^*(B\tilde{G})\\&\to H^0(\mathbf{Z},H^*(B(S^1\times(\mathbf{Z}/m_1))))\to 0\end{aligned} \tag{8.6.27}$$

arising from the cohomology spectral sequence of (8.6.18) or, what amounts to the same thing, from the Wang sequence of a fibre bundle over S^1 modelling the extension (8.6.18); here we have again written $H^*(-)=H^*(-,\mathbf{Z})$ for short, and as before cohomology must be computed with respect to the induced action of the quotient group $\mathbf{Z}$ coming from the extension (8.6.18). The proof is now completed in much the same way as the proof of (8.4.8): Inspection shows that for degree reasons there is neither an additive nor a multiplicative extension problem in (8.6.27). Now a little thought reveals that the relations (8.6.9)–(8.6.17) are defining relations for the subalgebra of $H^*(\tilde{\mathrm{M}}(G))$ generated by the given generators; for example, the relations (8.6.11), (8.6.15), and (8.6.17) may be computed in $H(\mathrm{M}^*(S^1\times(\mathbf{Z}/m_1)))=H^*(B(S^1\times(\mathbf{Z}/m_1)),\mathbf{Z})$, and the relations (8.6.9), (8.6.12), (8.6.13), and (8.6.16) may be verified in formally the same way as (8.4.9), (8.4.12), and (8.4.13). The relation (8.6.14) is actually a consequence of the others, since the elements ζ_{2i} have infinite order. Further, it is clear that the image of this subalgebra under the restriction map

$$H^*(\tilde{\mathrm{M}}(G))\to H^*(B(S^1\times(\mathbf{Z}/m_1)),\mathbf{Z})^{\mathbf{Z}} \tag{8.6.28}$$

is the subalgebra generated by the classes c_1, lc_y^i, $i \geqslant 1$, and $(l/p)c_y^p - (l/p)\,k_1^{p-1}c_1^{p-1}c_y$, where $p \in P = \{p_1, ..., p_v\}$. We now prove that under the circumstances of (8.6.8) this subalgebra exhausts the invariants:

Notice that now also the classes $lc_y^i \in \mathrm{H}^*(B(S^1 \times (\mathbf{Z}/m_1)), \mathbf{Z})$, $i \geqslant 1$, come into play; however, their images in $\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})$ under the obvious map are trivial. To understand their significance a bit better, we assume for the moment that l and k_1 are arbitrary, and we proceed in somewhat greater generality than actually needed at the moment; in the next subsection it will enable us to obtain some insight into the structure of the invariants and of the cohomology rings of G and $B\tilde{G}$ in the general case.

Consider the exact sequence

$$0 \to l(\mathrm{P}[c_y, c_1]/m_1c_1) \to (\mathrm{P}[c_y, c_1]/m_1c_1) \to \mathrm{P}_{(\mathbf{Z}/l)}[c_y, c_1] \to 0. \tag{8.6.29}$$

Taking invariants, we obtain at once the following.

LEMMA 8.6.30. *The invariants*

$$(\mathrm{H}^*(B(S^1 \times \mathbf{Z}/m_1), \mathbf{Z}))^{\mathbf{Z}} = (\mathrm{P}[c_y, c_1]/m_1c_1)^{\mathbf{Z}}$$

fit into an exact sequence

$$\begin{aligned} 0 &\longrightarrow l\mathrm{P}[c_y, c_1]/m_1c_1 \longrightarrow (\mathrm{P}[c_y, c_1]/m_1c_1)^{\mathbf{Z}} \\ &\longrightarrow (\mathrm{P}_{(\mathbf{Z}/l)}[c_y, c_1])^{\mathbf{Z}} \xrightarrow{\delta} \mathrm{H}^1(\mathbf{Z}/l, l\mathrm{P}[c_y, c_1]/m_1c_1) \longrightarrow \cdots. \end{aligned} \tag{8.6.31}$$

In fact, the left-hand term in (8.6.29) is already invariant, and (8.6.31) is just the beginning of the corresponding long exact cohomology sequence with $\mathbf{Z}$ replaced by $\mathbf{Z}/l$, which is all right since the action factors through $\mathbf{Z}/l$. ∎

The proof of (8.6.8) is now completed by the following.

PROPOSITION 8.6.32. *Under the circumstances of* (8.4.8) *the subalgebra of invariants is generated as an algebra by the classes* c_1, lc_y^i, $i \geqslant 1$, *and* $(l/p)c_y^p - (l/p)\,k_1^{p-1}c_1^{p-1}c_y$, *where* $p \in P = \{p_1, ..., p_v\}$.

Proof. It is easy to see that, since k_1 and l are assumed relatively prime, the boundary map δ in (8.6.31) is zero. The left-hand term in (8.6.31) now provides the classes lc_y^i. The proof is then completed in virtually the same way as the one for (8.4.27) above. We leave the details to the reader. ∎

The relation (8.4.14) now follows at once from Lemma 8.5.24 and the relation (8.6.14), since by construction the morphism (8.6.6) is multiplicative and sends the elements $\tilde{\xi}_{2i+1}$ and $\tilde{\zeta}_{2i}$ to ξ_{2i+1} and ζ_{2i}, respectively, with the understanding that $\zeta_2 = 0 \in \mathrm{H}^2(G, \mathbf{Z})$.

Remark 8.6.33. From (8.4.4) we have $d(\omega_y) = -\tilde{\zeta}_2$. Hence $-\tilde{\zeta}_2$ may be viewed as the characteristic class of (8.6.5).

Remark 8.6.34. Since for degree reasons the extension problem for (8.6.27) is trivial, the computation of the cohomology of $B\tilde{G}$ does not require any new machinery, and the reader might get the impression that one could therefore compute the cohomology of G by means of the Gysin sequence of (8.6.5). However, this Gysin sequence comes with an extension problem. While above we did not exploit the Gysin sequence, if one did compute the cohomology of G by means of the Gysin sequence, our construction would provide the solution of the extension problem.

8.7. *Invariant Theory and More Classes*

It is easy to see that without the hypothesis in (8.4.8) the graded commutative algebra given there by generators and relations is actually a subalgebra of the integral cohomology ring of G. In view of the exactness of (8.5.2), the reason why under the circumstances of (8.4.8) this subalgebra exhausts the entire cohomology ring is given in (8.5.27). In the present subsection we shall indicate how more classes can be constructed if the hypothesis of (8.4.8) does not hold. This will justify our claim made in the Introduction of [49] that as far as the present example is concerned, our approach reduces the computation of $\mathrm{H}^*(G, \mathbf{Z})$ to that of the invariants in $\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})$: Let J be a homogeneous polynomial in c_y and c_1 which is an invariant in

$$\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}) = (\mathrm{P}[c_y, c_1] \otimes \Lambda[\omega_3])/(m_1 c_1, l c_y, l\omega_3).$$

Then $d_1(J) = 0$, where d_1 is the operator introduced in (8.5), whence there is a polynomial $p_J(c_y, c_1)$ so that in

$$\begin{aligned}(\bar{\mathrm{M}}(G))^* &= \mathrm{P}[c_1] \otimes_{\vartheta^*} (\bar{\mathrm{M}}_G)^* \\ &= \mathrm{P}[c_1] \otimes_{\vartheta^*} (\Lambda[\omega_1, \omega_2] \otimes_{\xi^*} (\mathrm{P}[c_y] \otimes_{\tau_N^*} \Lambda[\omega_y]))\end{aligned}$$

the relation

$$dJ = d(p_J(c_y, c_1))\,\omega_2$$

holds. Then $\sigma_J = J - p_J(c_y, c_1)\omega_2$ is a cocycle in $(\bar{\mathrm{M}}(G))^*$ which by construction represents a pre-image in $\mathrm{H}^*(G, \mathbf{Z})$ of the class of J in $(\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z}))^{\mathbf{Z}}$. Special cases of this are the classes σ_{2p} introduced in (8.4.7) above. It is clear that if $J_1, J_2, \cdots$ is a set of generators of the subalgebra of invariants in $\mathrm{P}[c_y, c_1]/(m_1 c_1, l c_y)$, the cohomology ring $\mathrm{H}^*(G, \mathbf{Z})$ is as an algebra generated by the corresponding classes $\sigma_1, \sigma_2, \cdots$ together with the classes $c_1, \xi_1, \zeta_3, ..., \omega_3$. For example, it is known (Steinberg [59]) that if the group $\mathbf{Z}/p^r = \langle x; x^{p^r} = 1 \rangle$ acts on the algebra $(\mathbf{Z}/p^r[z_1, z_2])$ by means of

$$x \cdot z_1 = z_1, \qquad x \cdot z_2 = z_1 + z_2,$$

the algebra of invariants $(\mathbf{Z}/p^r[z_1, z_2])^{\mathbf{Z}/p^r}$ is as an algebra generated by z_1 and the polynomials $p^j(z_2^p - z_1^{p-1} z_2)^i$ having the property that the number $p^j i$ is divisibel by p^{r-1}. We do not know whether a set of defining relations for this algebra of invariants has ever been worked out.

To indicate how messy the situation may be in general, we mention that if l divides k_1, as an algebra, the subalgebra $(\mathrm{H}^*(B(S^1 \times \mathbf{Z}/m_1), \mathbf{Z}))^{\mathbf{Z}}$ of invariants is generated by c_1 and the classes $(l/l\{j\}) c_y^j$, where $1 \leqslant j \leqslant l$. This may be deduced from the exactness of (8.6.31). Pre-images in $\mathrm{H}^*(\tilde{\mathrm{M}}(G))$ of these classes are the classes $\tilde{\zeta}_{2j}$. However, it is manifest that the subalgebra of invariants in $\mathrm{H}^*((\mathbf{Z}/l) \times (\mathbf{Z}/m_1), \mathbf{Z})$ will be generated by $c_1, \omega_3, c_y^2, c_y^3, \ldots,$ and not just by $c_1, \omega_3, \zeta_2, \zeta_3$, etc., and the integral cohomology ring of G has rather a weird structure.

8.8. *A Counterexample*

As explained in (3.0) of [49], when Q is finitely generated and the construction of

$$\mathrm{M}(Q) = \Gamma_L \otimes_{\iota_Q} A(Q)$$

is carried out as in [49, (3.0.8)], the diagonal map on $\mathrm{M}(Q)$ passes to a coalgebra structure on the reduced object $\bar{\mathrm{M}}(Q)$; cf. [49, (3.0.9)]. This raises the question whether under these circumstances $\bar{\mathrm{M}}(G)$ way be written as a bundle

$$\bar{\mathrm{M}}(G) = \bar{\mathrm{M}}(Q) \otimes_\tau \bar{\mathrm{M}}(N)$$

for an appropriate twisting cochain $\tau\colon \bar{\mathrm{M}}(Q) \to \bar{\mathrm{M}}(N)$. We now indicate briefly that under the circumstances of (8.4)–(8.7) this is the case if and only if $m_1 = l$. To this end, we recall from (3.0.7)–(3.0.9) of [49] that the inherited diagonal map on the above reduced object $\bar{\mathrm{M}}(Q)$ is given by the rules

$$\Delta(v_i) = v_i \otimes 1 + 1 \otimes v_i, \qquad i = 1, 2,$$

$$\Delta(u_1) = u_1 \otimes 1 + 1 \otimes u_1 + \frac{m_1(m_1 - 1)}{2} v_1 \otimes v_1;$$

hence the graded module underlying $\bar{\mathrm{M}}(G)$ admits the structure of an induced comodule

$$\Delta\colon \bar{\mathrm{M}}(G) \to \bar{\mathrm{M}}(Q) \otimes \bar{\mathrm{M}}(G)$$

over the underlying graded coalgebra of $\bar{M}(Q)$. Since by construction the differential d is compatible with the obvious differential graded $\bar{M}(N)$-module structure, in view of a theorem of Gugenheim [16] (cf. also Gugenheim [17], where this theorem has been rephrased as (2.2), $(2.4)_*$, and $(2.4)^*$), the object $\bar{M}(G)$ may be written as a bundle as desired if and only if d is compatible with the induced comodule structure. We now ask when this is the case.

Recall that, in view of (8.4.1),

$$d(u_1 v_2) = k_1 \left(u_y - \frac{l(l-1)}{2} v_1 v_y \right) + m_1 v_1 v_2.$$

Now

$$\begin{aligned} \Delta(u_1 v_2) &= u_1 v_2 \otimes 1 + u_1 \otimes v_2 + v_2 \otimes u_1 + 1 \otimes u_1 v_2 \\ &\quad + \frac{m_1(m_1-1)}{2} (v_1 \otimes v_1 v_2 - v_1 v_2 \otimes v_1), \end{aligned}$$

whence

$$\begin{aligned} &(d \otimes \bar{M}(G) + \bar{M}(Q) \otimes d)\, \Delta(u_1 v_2) \\ &= m_1(v_1 v_2 \otimes 1 + v_1 \otimes v_2 - v_2 \otimes v_1) \\ &\quad + 1 \otimes d(u_1 v_2) - \frac{m_1(m_1-1)}{2} v_1 \otimes v_y \\ &= k_1 \otimes u_y - \frac{m_1(l-1)}{2} \otimes v_1 v_y - \frac{m_1(m_1-1)}{2} v_1 \otimes v_y \\ &\quad + m_1(v_1 v_2 \otimes 1 + v_1 \otimes v_2 - v_2 \otimes v_1 + 1 \otimes v_1 v_2). \end{aligned}$$

On the other hand, in view of the formula for $d(u_1 v_2)$ given in (8.4.1) above,

$$\begin{aligned} \Delta(d(u_1 v_2)) &= k_1 \otimes u_y - \frac{m_1(l-1)}{2} (v_1 \otimes v_y + 1 \otimes v_1 v_y) \\ &\quad + m_1(v_1 v_2 \otimes 1 + v_1 \otimes v_2 - v_2 \otimes v_1 + 1 \otimes v_1 v_2). \end{aligned}$$

This shows that the differential d is at most compatible with the induced comodule structure if $m_1 = l$. On the other hand, if $m_1 = l$, the rules

$$\begin{aligned} \tau(\gamma_i(u_1)) &= 0 \\ \tau((\gamma_i(u_1)) v_1) &= 0 \\ \tau((\gamma_i(u_1)) v_2) &= -\gamma_i(u_y) \\ \tau((\gamma_i(u_1))\, v_1 v_2) &= -(\gamma_i(u_y)) v_y \end{aligned}$$

yield a twisting cochain $\tau: \bar{\mathrm{M}}(Q) \to \bar{\mathrm{M}}(N)$ so that indeed $\bar{\mathrm{M}}(G) = \bar{\mathrm{M}}(Q) \otimes_\tau \bar{\mathrm{M}}(N)$. It is easy to see this, and we leave the details to the reader.

It is perhaps worthwhile mentioning that a related example of a construction which is not a bundle is given on page 186 of Moore [36].

8.9. *The Case* $m_1 = 0$, $m_2 > 1$

Now $Q = \mathbf{Z} \times (\mathbf{Z}/m_2)$, and the recipe for the twisting cochain

$$\vartheta: \Gamma[u_2] \to \mathrm{end}_{\mathrm{M}(N) \otimes_{RN} RG} (\mathrm{M}_G)$$

given in (4.3.5) of [49] yields the following formula (8.9.1) for ϑ, where the notation is that introduced in (8.2):

$$\theta(\gamma_1(u_2)) = A_2(U_2 + U_{2,1}), \qquad \theta(\gamma_i(u_2)) = A_2^i U_{2,i}, \qquad i \geqslant 2. \quad (8.9.1)$$

The property (5.3) in order for ϑ to be a twisting cochain now comes down to the following commutation rules in RG:

$$\begin{aligned} (y-1)\, D_2 &= b_2 x_1 - x_1 b_2 \\ x_1 x_2 b_2 + (x_2 - 1)\, D_2 &= Y x_1 x_2^l \\ b_2 x_1 x_2 + D_2(x_2 - 1) &= Y x_1 x_2^l. \end{aligned} \qquad (8.9.2)$$

Notice these are entirely parallel to (8.3.4). As far as cohomology is concerned, nothing new is added here to what was already said in (8.3)–(8.7).

For completeness we give the corresponding reduced operators

$$\vartheta(\gamma_i(u_2)) \in \mathrm{end}_{\bar{\mathrm{M}}(N)} (\bar{\mathrm{M}}_G), \qquad i \geqslant 1,$$

since we shall need them in the next subsection. They look like

$$\vartheta(\gamma_1(u_2)) = k_2 \begin{bmatrix} 0 & \gamma_1(u_y) & 0 & -(\gamma_1(u_y))v_y \\ 0 & 0 & 0 & 0 \\ -l & -\dfrac{l(l-1)}{2} v_y & 0 & \gamma_1(u_y) \\ 0 & l & 0 & 0 \end{bmatrix},$$

$$\vartheta(\gamma_i(u_2)) = (-1)^{i+1} k_2^i \begin{bmatrix} 0 & \gamma_i(u_y) & 0 & -(\gamma_i(u_y))v_y \\ 0 & 0 & 0 & 0 \\ 0 & -\dfrac{l(l-1)}{2} (\gamma_{i-1}(u_y))v_y & 0 & \gamma_i(u_y) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$i \geqslant 2.$

Hence, with reference to the differential graded $\bar{\mathrm{M}}(N)$-module structure on $\bar{\mathrm{M}}_G$, the differential d on $\bar{\mathrm{M}}(G)$ may explicitly be described by the formulas

$$d(\gamma_i(u_2)) = m_2((\gamma_{i-1}(u_2))v_2),$$

$$d((\gamma_i(u_2))v_1) = \begin{cases} \sum_{j=1}^{i} \left((-k_2)^j (\gamma_{i-j}(u_2)) \left(\gamma_j(u_y) - \frac{l(l-1)}{2} v_2(\gamma_{j-1}(u_y))v_y\right)\right) \\ -m_2((\gamma_{i-1}(u_2))\, v_1 v_2) \end{cases} \tag{8.9.3}$$

$$d((\gamma_i(u_2))v_2) = 0,$$

$$d((\gamma_i(u_2))\, v_1 v_2) = \begin{cases} -(\gamma_i(u_2))v_y \\ -\sum_{j=1}^{i} (-k_2)^j (\gamma_{i-j}(u_2))((-\gamma_j(u_y))v_y + v_2(\gamma_j(u_y))), \end{cases}$$

where $i \geqslant 1$ or $i \geqslant 0$ as appropriate.

8.10. *The Case* $m_1 > 1$, $m_2 > 1$

Now $Q = (\mathbf{Z}/m_1) \times (\mathbf{Z}/m_2)$, and $G = G(m_1, m_2, l)$ is finite of order $lm_1 m_2 = l^3 k_1 k_2$. Before we go into details we mention that the integral cohomology rings of the groups $G(p, p, p)$ for an odd prime p have been calculated by Lewis [52]. Notice that the group $G(2, 2, 2)$ is the dihedral group of order eight which is sort of an easy case, since this group may be written as the wreath product of $\mathbf{Z}/2$ with $\mathbf{Z}/2$. It is also interesting to observe that the group $G(p, p, p)$ can be realized as the unipotent radical in $GL(3, \mathbf{F}_p)$, for example, as the subgroup of upper triangular matrices with 1's on the diagonal.

As before the recipe in (4.3.5) of [49] yields the requisite twisting cochain

$$\vartheta : \Gamma[u_1, u_2] \to \operatorname{end}_{\mathrm{M}(N) \otimes_{RN} RG} (\mathrm{M}_G).$$

In view of the naturality of the construction, from (8.3.1) and (8.9.1) we can deduce that

$$\vartheta(\gamma_1(u_i)) = A_i(U_i + U_{i,1}),\ i = 1, 2, \qquad \vartheta(\gamma_k(u_i)) = A_i^k U_{i,k},\ i = 1, 2, k \geqslant 2. \tag{8.10.1}$$

The computation of the remaining terms $\vartheta((\gamma_a(u_1))(\gamma_b(u_2)))$, $a \neq 0 \neq b$, is straightforward, but the amount of labor involved is prohibitive. So far we have been unable to compute these terms in a closed form, and specific calculations are perhaps more conveniently carried out on a computer using some formal manipulations machinery. However, since ϑ is a twisting cochain, we can deduce some information about the reduced operators

$$\vartheta((\gamma_a(u_1))(\gamma_b(u_2))) \in \operatorname{end}_{\bar{\mathrm{M}}(N)}(\bar{\mathrm{M}}_G), \qquad a \neq 0 \neq b,$$

in an entirely formal way. For example, in view of (5.3) we must have

$$D^{\xi}(\vartheta(u_1 u_2)) = (\vartheta(u_1))(\vartheta(u_2)) + (\vartheta(u_2))(\vartheta(u_1))$$

$$= k_1 k_2 \begin{bmatrix} 0 & \frac{l(l+1)}{2} u_y v_y & \frac{l(l+1)}{2} u_y v_y & 0 \\ -\frac{l^2(l-1)}{2} v_y & 0 & 0 & \frac{l(l+1)}{2} u_y v_y \\ \frac{l^2(l-1)}{2} v_y & 0 & 0 & -\frac{l(l+1)}{2} u_y v_y \\ 0 & \frac{l^2(l-1)}{2} v_y & \frac{l^2(l-1)}{2} v_y & 0 \end{bmatrix},$$

where D^{ξ} denotes the twisted differential in $\operatorname{end}_{\bar{\mathrm{M}}(N)}(\bar{\mathrm{M}}_G)$. Now for an odd degree operator $\Theta = [\Theta_{i,j}] \in \operatorname{end}_{\bar{\mathrm{M}}(N)}(\bar{\mathrm{M}}_G)$, this differential is given by the formula

$$D^{\xi}(\Theta) = \begin{bmatrix} d\Theta_{1,1} & d\Theta_{1,2} - v_y \Theta_{4,2} & d\Theta_{1,3} - v_y \Theta_{4,3} & d\Theta_{1,4} \\ -d\Theta_{2,1} & -d\Theta_{2,2} & -d\Theta_{2,3} & -d\Theta_{2,4} - \Theta_{2,1} v_y \\ -d\Theta_{3,1} & -d\Theta_{3,2} & -d\Theta_{3,3} & -d\Theta_{3,4} - \Theta_{3,1} v_y \\ d\Theta_{4,1} & d\Theta_{4,2} & d\Theta_{4,3} & d\Theta_{4,4} \end{bmatrix}, \tag{8.10.2}$$

where d refers to the differential in $\bar{\mathrm{M}}(N)$. It is not hard to see that this implies

$$\vartheta(u_1 u_2) = k_1 k_2 \begin{bmatrix} \vartheta_{1,1} & l\gamma_2(u_y) & l\gamma_2(u_y) & \vartheta_{1,4} \\ \frac{l(l-1)}{2} u_y & \vartheta_{2,2} & \vartheta_{2,3} & -l\gamma_2(u_y) \\ -\frac{l(l-1)}{2} u_y & \vartheta_{3,2} & \vartheta_{3,3} & l\gamma_2(u_y) \\ \vartheta_{4,1} & \frac{l(l-1)}{2} u_y & \frac{l(l-1)}{2} u_y & \vartheta_{4,4} \end{bmatrix},$$

where the terms labelled $\vartheta_{*,*}$ indicate possibly trivial odd degree entries from $\bar{\mathrm{M}}(N)$. At present we do not know whether we actually have to compute all these entries, since it may be that, as far as cohomology is concerned, some of them do *not* contain any new information. We shall see below that, e.g., for the dihedral group of order eight they are all inessential.

We now illustrate our approach with a number of sample calculations of some homology and cohomology groups of the groups $G=G(m_1, m_2, l)$, thereby demonstrating that explicit calculations are now possible. To this end we write out those terms of the differential on the resulting reduced object

$$\begin{aligned}\bar{\mathrm{M}}(G) &= \Gamma[u_1, u_2] \otimes_{\vartheta} \bar{\mathrm{M}}_G \\ &= \Gamma[u_1, u_2] \otimes_{\vartheta} (\Lambda[v_1, v_2] \otimes_{\xi} (\Gamma[u_y] \otimes_{\tau_N} \Lambda[v_y]))\end{aligned}$$

of the kind [49, (4.3.10)] which may be obtained from the calculations that we have carried out above. In view of the naturality of the construction, the calculation of the reduced operators $\vartheta(\gamma_i(u_1))$ and $\vartheta(\gamma_i(u_2))$ in (8.4) and (8.9) also yields these operators under the present circumstances. Keeping in mind what was said above about $\vartheta(u_1 u_2)$, we thus obtain, with reference to the differential graded $\bar{\mathrm{M}}(N) = (\Gamma[u_y] \otimes_{\tau_N} \Lambda[v_y])$-structure on $\bar{\mathrm{M}}(G)$, the formulas

$$d(v_1) = 0 \tag{8.10.3}$$

$$d(v_2) = 0 \tag{8.10.4}$$

$$d(v_1 v_2) = -v_y \tag{8.10.5}$$

$$d(\gamma_i(u_1)) = m_1 \gamma_{i-1}(u_1) v_1 \tag{8.10.6}$$

$$d(\gamma_i(u_2)) = m_2 \gamma_{i-1}(u_2) v_2 \tag{8.10.7}$$

$$d(\gamma_i(u_1) v_1) = 0 \tag{8.10.8}$$

$$d(\gamma_i(u_2) v_2) = 0 \tag{8.10.9}$$

$$d(u_1 v_2) = k_1 \left(u_y - \frac{l(l-1)}{2} v_1 v_y \right) + m_1 v_1 v_2 \tag{8.10.10}$$

$$d(u_2 v_1) = -k_2 \left(u_y - \frac{l(l-1)}{2} v_2 v_y \right) - m_2 v_1 v_2 \tag{8.10.11}$$

$$d(u_1 u_2) = \left\{ \begin{array}{c} m_2 u_1 v_2 + m_1 u_2 v_1 \\ -k_1 k_2 \dfrac{l(l-1)}{2} v_1 u_y + k_1 k_2 \dfrac{l(l-1)}{2} v_2 u_y \\ + ? u_y v_y + ? v_1 v_2 v_y \end{array} \right. \tag{8.10.12}$$

$$d(u_1 v_1 v_2) = -u_1 v_y - k_1 (u_y v_y + v_1 u_y) \tag{8.10.13}$$

$$d(u_2 v_1 v_2) = -u_2 v_y + k_2 (u_y v_y - v_2 u_y) \tag{8.10.14}$$

$$d(\gamma_2(u_1)v_2) = \begin{cases} k_1(\gamma_1(u_1))\left(\gamma_1(u_y) - \frac{l(l-1)}{2} v_1 v_y\right) \\ \qquad + k_1^2\left(\gamma_2(u_y) - \frac{l(l-1)}{2} v_1((\gamma_1(u_y))v_y\right) \\ \qquad + m_1(\gamma_1(u_1))\, v_1 v_2 \end{cases} \tag{8.10.15}$$

$$d(u_1 u_2)v_1 = u_1(d(u_2 v_1)) - k_1 k_2 l \gamma_2(u_y) - k_1 k_2 \frac{l(l-1)}{2} v_1 v_2 u_y + ?v_1 + ?v_2 \tag{8.10.16}$$

$$d(u_1 u_2)v_2 = u_2(d(u_1 v_2)) - k_1 k_2 l \gamma_2(u_y) - k_1 k_2 \frac{l(l-1)}{2} v_1 v_2 u_y + ?v_1 + ?v_2 \tag{8.10.17}$$

$$d(\gamma_2(u_2)v_1) = \begin{cases} -k_2(\gamma_1(u_2))\left(\gamma_1(u_y) - \frac{l(l-1)}{2} v_2 v_y\right) \\ \qquad + k_2^2\left(\gamma_2(u_y) - \frac{l(l-1)}{2} v_2((\gamma_1(u_y))v_y\right) \\ \qquad - m_2(\gamma_1(u_2))\, v_1 v_2. \end{cases} \tag{8.10.18}$$

Notice that (8.10.3)–(8.10.9) are not especially interesting since they do not contain any new information, while in (8.10.10)–(8.10.18) interesting new terms show up.

To get some insight into the cohomology of G, we dualize (8.10.3)–(8.10.18) and obtain the following formulas for the differential as a morphism of graded $\mathrm{P}[c_1, c_2]$-modules, where we write $c_1' = k_1 c_1$, $c_2' = k_2 c_2$ for convenience:

$$d(\omega_1) = l c_1' \tag{8.10.19}$$

$$d(\omega_2) = l c_2' \tag{8.10.20}$$

$$d(\omega_y) = \omega_1 \omega_2 + l c_y \tag{8.10.21}$$

$$d(c_y) = c_2' \omega_1 - c_1' \omega_2 \tag{8.10.22}$$

$$d(\omega_1 \omega_2) = l(c_1' \omega_2 - c_2' \omega_1) \tag{8.10.23}$$

$$d(\omega_1 \omega_y) = l c_1' \omega_y - l \omega_1 c_y - \frac{l(l-1)}{2} c_1' \omega_2 \tag{8.10.24}$$

$$d(\omega_2 \omega_y) = l c_2' \omega_y - l \omega_2 c_y + \frac{l(l-1)}{2} c_2' \omega_1 \tag{8.10.25}$$

$$d(\omega_1 c_y) = l c_1' c_y + \omega_1 \omega_2 c_1' - \frac{l(l-1)}{2} c_1' c_2' \tag{8.10.26}$$

$$d(\omega_2 c_y) = lc'_2 c_y + \omega_1\omega_2 c'_2 + \frac{l(l-1)}{2} c'_1 c'_2 \tag{8.10.27}$$

$$d(c_y\omega_y) = \omega_1\omega_2 c_y + lc_y^2 + (c'_2\omega_1 - c'_1\omega_2)\omega_y + (c'_1 - c'_2)\,\omega_1\omega_2 + ?c'_1 c'_2 \tag{8.10.28}$$

$$d(\omega_1\omega_2\omega_y) = l(c'_1\omega_2\omega_y - c'_2\omega_1\omega_y + \omega_1\omega_2 c_y) + ?c'_1 c'_2 \tag{8.10.29}$$

$$d(c_y^2) = 2(c'_2\omega_1 - c'_1\omega_2)c_y - (c'_1)^2\omega_2 - (c'_2)^2\omega_1 + lc'_1 c'_2(\omega_1 + \omega_2). \tag{8.10.30}$$

Hence $\mathrm{H}^1(G, \mathbf{Z})$ is zero, and $\mathrm{H}^2(G, \mathbf{Z})$ is generated by c_1 and c_2, subject to the relations

$$m_1 c_1 = 0, \qquad m_2 c_2 = 0. \tag{8.10.31}$$

Furthermore,

$$\xi_1 = (c'_1 + c'_2)\omega_y - (\omega_1 + \omega_2)c_y,$$
$$\xi_2 = (c'_1 - c'_2)\omega_y - (\omega_1 - \omega_2)c_y - (l-1)\,c'_1\omega_2,$$
$$\omega_3 = \frac{k_2}{(k_1, k_2)} c_2\omega_1 - \frac{k_1}{(k_1, k_2)} c_1\omega_2$$

are 3-cocycles, and from

$$d\left((\omega_1 + \omega_2)\omega_y - \frac{l(l-1)}{2} c_y\right) = l\xi_1,$$
$$d\left((\omega_1 - \omega_2)\omega_y + \frac{l(l-1)}{2} c_y\right) = l\xi_2,$$
$$(l-1)\,d(\omega_1\omega_y) = \frac{l(l-1)}{2}(\xi_1 + \xi_2),$$
$$(l-1)\,d\left(\omega_2\omega_y - \frac{l(l-1)}{2} c_y\right) = \frac{l(l-1)}{2}(\xi_1 - \xi_2),$$
$$d(c_y) = (k_1, k_2)\omega_3,$$

we see that

$$l\xi_1 = 0 \in \mathrm{H}^3(G, \mathbf{Z}), \tag{8.10.32}$$
$$l\xi_2 = 0 \in \mathrm{H}^3(G, \mathbf{Z}), \tag{8.10.33}$$
$$\frac{l(l-1)}{2}\xi_1 = \frac{l(l-1)}{2}\xi_2 \in \mathrm{H}^3(G, \mathbf{Z}), \tag{8.10.34}$$
$$(k_1, k_2)\omega_3 = 0 \in \mathrm{H}^3(G, \mathbf{Z}), \tag{8.10.35}$$

where we do not distinguish in notation between cocycles and classes they represent. Likewise, from (8.10.29) we deduce that

$$\chi_4 = c'_1 \omega_2 \omega_y - c'_2 \omega_1 \omega_y + \omega_1 \omega_2 c_y$$

is a 4-cocycle, and that

$$l\chi_4 = ?c'_1 c'_2; \tag{8.10.36}$$

to compute ? we must determine the coefficient $\vartheta_{4,4}$ in $\vartheta(u_1 u_2)$. Similarly,

$$d(\omega_1 c_y - c'_1 \omega_y) = -\frac{l(l-1)}{2} c'_1 c'_2,$$

whence

$$\frac{l(l-1)}{2} k_1 k_2 c_1 c_2 = 0 \in \mathrm{H}^4(G, \mathbf{Z}). \tag{8.10.37}$$

In particular, if l is odd, (8.10.37) is vacuous, while if l is even, it says

$$\frac{l}{2} k_1 k_2 c_1 c_2 = 0 \in \mathrm{H}^4(G, \mathbf{Z}).$$

It is clear that (8.10.31) are defining relations for $\mathrm{H}^2(G, \mathbf{Z})$. To see that the above yield defining relations for $\mathrm{H}^3(G, \mathbf{Z})$, we argue as follows: Filtering $\tilde{\mathrm{M}}(G)$ by Γ_L-degree yields a spectral sequence (E_r, d_r) with $E_2 = \mathrm{P}[c_1, c_2] \otimes \mathrm{H}^*(G_F, \mathbf{Z})$; here G_F and L are the corresponding groups explained in (5.1), and G is the quotient of a central extension

$$0 \to L \to G_F \to G \to 1,$$

where L is free abelian, generated by $x_1^{m_1} \in G_F$ and $x_2^{m_2} \in G_F$. We shall show elsewhere that this spectral sequence may be identified with the cohomology spectral sequence of the fibration

$$K(G_F) \to K(G, 1) \to K(L, 2);$$

however, this is *not* of any significance for what we are about to say. The cohomology ring of the group

$$G_F = G(0, 0, l) = \langle x_1, x_2, y; [x_2, x_1] = y, [x_1, y] = 1, [x_2, y] = 1, y^l = 1 \rangle$$

has been given in (7.3). Over the integers it is generated by classes ω_1 and ω_2 of degree 1 and a class c_y of degree 2, subject to the relations

$$\omega_1^2 = 0, \omega_2^2 = 0, \omega_1 \omega_2 = -lc_y, l\omega_1 c_y = 0, l\omega_2 c_y = 0, l\omega_1 \omega_2 c_y = 0. \tag{8.10.38}$$

We mention that in the cochain picture, ω_1, ω_2, and c_y are the duals of respectively v_1, v_2, and u_y in the bases of monomials. Inspection of

(8.10.19)–(8.10.22) shows that, as a $P[c_1, c_2] = H^*(L, 2)$-module morphism, the differential d_2 satisfies the formulas

$$d_2(\omega_1) = m_1 c_1, \qquad d_2(\omega_2) = m_2 c_2, \qquad d_2(c_y) = c_2' \omega_1 - c_1' \omega_2, \quad (8.10.39)$$

and we see at once that $(k_2/(k_1, k_2))\, c_2 \omega_1 - (k_1/(k_1, k_2))\, c_1 \omega_2$ is an infinite 3-cycle of order (k_1, k_2). Hence $\omega_3 \in H^3(G, \mathbf{Z})$ has the asserted order. Moreover, in view of (8.10.38), $\omega_1 c_y$ and $\omega_2 c_y$ restrict to classes in $H^3(G_F, \mathbf{Z})$ of order l, and, in view of (8.10.26) and (8.10.27), the first possibly non-zero differentials on these elements are given by

$$d_4(\omega_1 c_y) = -\frac{l(l-1)}{2} c_1' c_2', \qquad d_4(\omega_2 c_y) = \frac{l(l-1)}{2} c_1' c_2'. \quad (8.10.40)$$

Hence $(\omega_1 + \omega_2) c_y$ and $(\omega_1 - \omega_2) c_y$ are infinite cycles. The above classes ξ_1 and ξ_2 manifestly pass to $-(\omega_1 + \omega_2) c_y$ and $-(\omega_1 - \omega_2) c_y$, respectively, in the associated graded object. Hence (8.10.31)–(8.10.34) are in fact defining relations for $H^3(G, \mathbf{Z})$.

It is clear that $c_1^2 \in H^4(G, \mathbf{Z})$ and $c_2^2 \in H^4(G, \mathbf{Z})$ have orders m_1 and m_2, respectively. To get some more information about $H^4(G, \mathbf{Z})$, we observe that (8.10.29) implies

$$d(\omega_1 \omega_2 c_y) = l(c_1' \omega_2 - c_2' \omega_1) c_y - \frac{l(l-1)}{2} c_1' c_2' (\omega_1 + \omega_2). \quad (8.10.41)$$

Hence, if l is even, in view of (8.10.22), (8.10.30), and (8.10.41),

$$\zeta_4 = \frac{l}{2} (c_y^2 + (c_2' - c_1') c_y) + \omega_1 \omega_2 c_y$$

is a cocycle; moreover, in view of (8.10.21) and (8.10.28),

$$d(c_y \omega_y + (c_2' - c_1') \omega_y) = 2\zeta_4 - \chi_4 + ?\, c_1' c_2',$$

whence

$$2\zeta_4 = \chi_4 + ?\, c_1' c_2' \in H^4(G, \mathbf{Z}),$$

and to compute ? we must determine the entry $\vartheta_{1,1}$ in $\vartheta(u_1 u_2)$. In particular, the entries $\vartheta_{1,1}$ and $\vartheta_{4,4}$ determine the order of ζ_4. Notice that, in the above spectral sequence,

$$d_2(c_2' - c_1') c_y = c_1'^2 \omega_2 + c_2'^2 \omega_1 - c_1' c_2' (\omega_1 + \omega_2),$$

$$d_2(\omega_1 \omega_2 c_y) = 0,$$

$$d_4(\omega_1 \omega_2 c_y) = -\frac{l(l-1)}{2} c_1' c_2' (\omega_1 + \omega_2).$$

Further, if l is even,

$$d_2\left(\frac{l}{2}c_y^2\right)=0,$$

$$d_4\left(\frac{l}{2}c_y^2\right)=\frac{l}{2}\Big(-(c_1')^2\omega_2-(c_2')^2\omega_1+lc_1'c_2'(\omega_1+\omega_2)\Big),$$

whence

$$d_4\left(\frac{l}{2}c_y^2+\omega_1\omega_2c_y\right)=0,$$

i.e., $(l/2)c_y^2+\omega_1\omega_2c_y$ is an infinite cycle; it is clear that in the passage to the associated graded object the class ζ_4 goes to this infinite cycle.

We mention without proof that more calculations exhibit classes $\chi_{2i}\in \mathrm{H}^{2i}(G,\mathbf{Z})$, $i\geqslant 3$, enjoying properties similar to those of the class χ_4 constructed above, and, for each prime power p^k dividing l, a class $\zeta_{2p^k}\in \mathrm{H}^{2p^k}(G,\mathbf{Z})$ which does not lie in the subring generated by the other generators and has properties similar to the class ζ_4 constructed above. For example, the integral cohomology of the dihedral group $G(2,2,2)$ is generated by c_1, c_2, ξ_1, ζ_4, subject to the relations

$$2c_1=0,\ 2c_2=0,\ 4\zeta_4=0,\ c_1c_2=0,\ \xi_1^2=(c_1+c_2)\zeta_4.$$

This is of course well known. Notice that in view of (8.10.34) above, we have $\xi_1=\xi_2$, and there is *no* need to compute the entries $\vartheta_{*,*}$ in $\vartheta(u_1u_2)$ in this case. Likewise, Lewis [52] has shown, and it is not hard to reproduce this at the present stage from our calculations, that for p odd the integral cohomology of $G(p,p,p)$ is generated by $c_1, c_2, \xi_1, \xi_2, \chi_4, ..., \chi_{2p-2}, \zeta_{2p}$. To extend the known results about the dihedral group and the groups $G(p,p,p)$ to more general groups $G=G(m_1,m_2,l)$, we must at first compute more terms of the above twisting cochain ϑ; in view of a result of Evens [14], for a specific (finite) group G, it will always suffice to compute only a finite number of these terms. However, if we then wish to get explicit results, we run into the kind of problems related with invariant theory which have already been explained in (8.7). We intend to return to this elsewhere.

We mention in passing that, just as in (8.6) above, the obvious embedding of $\mathbf{Z}/l$ into the circle group S^1 gives rise to a non-connected Lie group $\tilde{G}$ which fits into a diagram of the kind (8.6.4), and there is again a Gysin sequence relating the cohomology of G and that of $B\tilde{G}$. Further, the terms in (8.10.18)–(8.10.29) which do not involve ω_y actually describe a small model for the cochains of $B\tilde{G}$ in low dimensions. An alternative approach to the cohomology of this space is to write it as the fibre of a map $B((\mathbf{Z}/m_1)\times(\mathbf{Z}/m_2)) \to K(\mathbf{Z}, 3)$ which represents the class ω_3, viewed as an element of $\mathrm{H}^3((\mathbf{Z}/m_1)\times(\mathbf{Z}/m_2), \mathbf{Z})$. Eventually, a complete calculation of the cohomology of G will presumably play off against each other the advantages of this construction and of the present approach. It is clear that, suitably rephrased, such a remark applies to an arbitrary nilpotent group of class two. Again we intend to come back to this elsewhere. We also mention that here a remark can be made similar to that in (8.6) referring to the role of the circle group as opposed to that of a Prüfer group.

Finally, let R be a ring of finite characteristic h, e.g., $R=\mathbf{Z}/p$ for a prime p. Then inspection of (5.3) and (8.10.1) shows that, for $i+j>0$, the reduced operators $\vartheta(\gamma_i(u_1)\,\gamma_j(u_2))$ may be written

$$\vartheta(\gamma_i(u_1)\,\gamma_j(u_2)) = k_1^i k_2^j \Theta(i, j) \in \mathrm{end}_{\bar{\mathrm{M}}(N)}(\bar{\mathrm{M}}_G)$$

for appropriate operators $\Theta(i, j)$. Hence, if the numbers k_1 and k_2 are divisible by h,

$$\mathrm{H}^*(G, R) = \mathrm{P}[c_1, c_2] \otimes \mathrm{H}^*(G(0, 0, l), R),$$

while if k_2 is divisible by h but not necessarily k_1,

$$\mathrm{H}^*(G, R) = \mathrm{P}[c_2] \otimes \mathrm{H}^*(G(m_1, 0, l), R).$$

Notice the groups $G(0, 0, l)$ and $G(m_1, 0, l)$ have been dealt with in (7.3) and (8.4) above. Thus the most interesting case occurs if k_1 and k_2 are not divisible by h, while h and l are not relatively prime. Examples are the groups $G(p, p, p)$ for a prime p already mentioned above. It is clear that from the results of Lewis [52] about the integral cohomology of these groups, in the standard way, the additive structure of the mod p cohomology may be derived. Moreover, a kind of intuitive picture for the mod p cohomology rings of these groups has been given by Tezuka and Yagita [60] in terms of the variety defined by the even degree subring of the cohomology ring. However, apart from that very little in known about the mod p cohomology of the groups $G(m_1, m_2, l)$. In principle these groups are covered by our methods, but since we were so far unable to compute the above twisting cochain ϑ in a closed form, at present we can in general not complete the mod p calculation if k_1 and k_2 are both not divisible by p while l is, and our approach requires further research here.

In theory, our methods thus provide an understanding of the cohomology of nilpotent groups of class 2, and what remain are more calculations.

Acknowledgments

I am indebted to J. Stasheff for his comments on an earlier version of the paper, which led to numerous improvements, and to R. Steinberg for discusssions about his work in invariant theory [59]. I owe a special debt of gratitude to S. Mac Lane for a number of most valuable comments on a draft of the paper. Most of this work was done during a stay at the Institute for Advanced Study, Princeton, New Jersey, and the paper was completed during a stay at the Mathematical Sciences Research Institute, Berkeley, California. I wish to express my gratitude for the hospitality of both institutions.

REFERENCES

References 1–42 are given in the first paper [49] of this series.

43. B. CENKL AND R. PORTER, Lazard completion of a group and free differential graded algebra models over subrings of the rationals, *Topology* **23** (1984), 445–464.
44. L. E. DICKSON, A fundamental system of invariants of the general modular linear group with a solution of the form problem, *Trans. Amer. Math. Soc.* **12** (1911), 75–98.
45. M. M. FELDSTEIN, Invariants of the linear group modulo p^k, *Trans. Amer. Math. Soc.* **25** (1923), 223–238.
46. L. GRUENENFELDER, On the homology of filtered and graded rings, *J. Pure Appl. Algebra* **14** (1979), 21–37.
47. V. K. A. M. GUGENHEIM AND J. D. STASHEFF, On perturbations and A_∞-structures, Festschrift in honor of G. Hirsch's 60th birthday (L. Lemaire, Ed.), *Bull. Soc. Math. Belg.* **38** (1986), 237–246.
48. M. HARADA AND A. KONO, On the integral cohomology rings of extraspecial 2-groups, *J. Pure Appl. Algebra* **44** (1987), 215–219.
49. J. HUEBSCHMANN, Perturbation theory and free resolutions for nilpotent groups of class 2, *J. Algebra* **126** (1989), 348–399.
50. L. LAMBE, Cohomology of principal G-bundles over a torus when $\mathrm{H}^*(BG; R)$ is polynomial, Festschrift in honor of G. Hirsch's 60th birthday (L. Lemaire, Ed.), *Bull. Soc. Math. Belg.* **38** (1986), 247–264.
51. L. LAMBE AND ST. PRIDDY, Cohomology of nilmanifolds and torsion free nilpotent groups, *Trans. Amer. Math. Soc.* **132** (1982), 39–55.
52. G. LEWIS, The integral cohomology rings of groups of order p^3, *Trans. Amer. Math. Soc.* **132** (1968), 501–529.
53. J. P. MAY, The cohomology of restricted Lie algebras and of Hopf algebras, *J. Algebra* **3** (1966), 123–146.
54. H. MUI, Modular invariant theory and cohomology algebras of symmetric groups, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **22** (1975), 319–369.
55. K. NOMIZU, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59** (1954), 531–538.
56. P. F. PICKEL, Rational cohomology of nilpotent groups and Lie algebras, *Comm. Algebra* **6** (1978), 409–419.
57. D. QUILLEN, On the associated graded ring of a group ring, *J. Algebra* **10** (1968), 410–418.
58. D. QUILLEN, The mod 2 cohomology rings of extra-special 2-groups and the spinor group, *Math. Ann.* **194** (1971), 197–223.
59. R. STEINBERG, On Dickson's theorem on invariants, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), 699–707.
60. M. TEZUKA AND N. YAGITA, The varieties of the mod p cohomology rings of extra special p-groups for an odd prime p, *Math. Proc. Cambridge Philos. Soc.* **94** (1983), 449–459.
61. J. S. TURNER, A fundamental system of invariants of a modular group of transformations, *Trans. Amer. Math. Soc.* **24** (1922), 129–134.